

INTRODUCTION TO MANY BODY PHYSICS: 620. Fall 2026

Answers to Questions I.

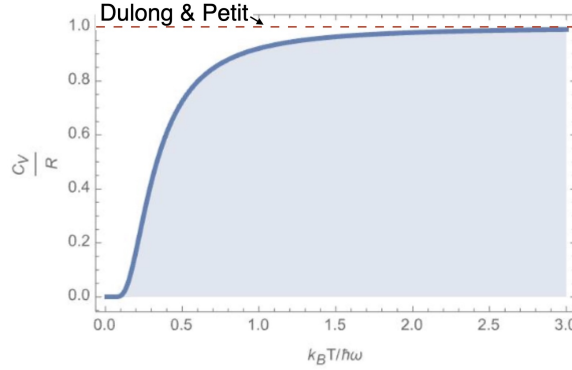


Figure 1: Plot of specific heat in the Einstein model of 1906.

1. The Hamiltonian for a single oscillator is

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \quad (1)$$

For an ensemble of N_{AV} oscillators, where N_{AV} is Avagadro's number, the expectation value of the energy is

$$E(T) = N_{AV}\langle H \rangle = N_{AV}\hbar\omega \left(n(\omega) + \frac{1}{2} \right) \quad (2)$$

where $n(\omega) = 1/(e^{\hbar\omega/k_B T} - 1)$. Differentiating to obtain the specific heat capacity

$$\begin{aligned} C_V &= \frac{dE}{dT} = N_{AV}\hbar\omega \frac{d}{dT} \left(\frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} \right) = N_{AV} \left(\frac{\hbar\omega}{k_B T^2} \right) \frac{e^{\frac{\hbar\omega}{k_B T}}}{(e^{\frac{\hbar\omega}{k_B T}} - 1)^2} \\ &= N_{AV}k_B \left[\left(\frac{\hbar\omega}{k_B T} \right)^2 \left(\frac{1}{2 \sinh\left(\frac{\hbar\omega}{2k_B T}\right)} \right)^2 \right] \\ &= R F \left[\frac{\hbar\omega}{k_B T} \right] \end{aligned} \quad (3)$$

where the function

$$F(x) = \left(\frac{x}{2 \sinh(x/2)} \right)^2$$

and $R = N_{AV}k_B$ is the gas constant (see Fig. 1.). Notice that at high temperatures $F[\hbar\omega/k_B T] \rightarrow 1$, so that $C_V(T) \rightarrow R$, as expected for two quadratic degrees of freedom, from the Dulong and Petit Law.

2. (a) Making the transformation

$$\begin{aligned} b &= ua + va^\dagger, \\ b^\dagger &= ua^\dagger + va, \end{aligned} \quad (4)$$

(where u and v are real), we find that

$$[b, b^\dagger] = [ua + va^\dagger, ua^\dagger + va] = u^2[a, a^\dagger] + v^2[a^\dagger, a] = u^2 - v^2, \quad (5)$$

and $[b, b] = [b^\dagger, b^\dagger] = 0$ trivially. If $u^2 - v^2 = 1$, then $[b, b^\dagger] = 1$ and the transformation is canonical.

(b) Let us assume that the Hamiltonian can be diagonalized in the form

$$H = \tilde{\omega}(b^\dagger b + \frac{1}{2}) = \frac{\tilde{\omega}}{2}(b^\dagger b + bb^\dagger). \quad (6)$$

Notice how I symmetrized the expression first. Substituting in the above transformation, we find that

$$\begin{aligned} (b^\dagger b + bb^\dagger) &= [(ua^\dagger + va)(ua + va^\dagger) + (ua + va^\dagger)(ua^\dagger + va)] \\ &= (u^2 + v^2)(a^\dagger a + aa^\dagger) + 2uv(a^2 + a^{\dagger 2}) \end{aligned} \quad (7)$$

Thus since $aa^\dagger = a^\dagger a + 1$

$$H = \omega(a^\dagger a + \frac{1}{2}) + \frac{1}{2}\Delta(a^\dagger a^\dagger + aa), \quad (8)$$

where

$$\omega = \tilde{\omega}(u^2 + v^2), \quad \Delta = \tilde{\omega}(2uv), \quad (9)$$

Squaring both terms, and subtracting we find

$$\begin{aligned} \omega^2 - \Delta^2 &= \tilde{\omega}^2[(u^2 + v^2)^2 - (2uv)^2] \\ &= \tilde{\omega}^2[u^4 + v^4 - 2(uv)^2] \\ &= \tilde{\omega}^2[u^2 - v^2]^2 \\ &= \tilde{\omega}^2, \end{aligned} \quad (10)$$

since $u^2 - v^2 = 1$, so that $\tilde{\omega} = \sqrt{\omega^2 - \Delta^2}$. Notice that the first condition in (9) forces $\tilde{\omega}$ to be formally positive. Physically, we do not expect a negative excitation energy! By substituting $v^2 = u^2 - 1$ into the first equation in (9), we obtain

$$\begin{aligned} u^2 &= \frac{1}{2}(1 + \omega/\tilde{\omega}) = \left(\frac{1}{2} + \frac{\omega}{2\sqrt{\omega^2 - \Delta^2}}\right), \\ v^2 &= -\frac{1}{2}(1 - \omega/\tilde{\omega}) = -\left(\frac{1}{2} - \frac{\omega}{2\sqrt{\omega^2 - \Delta^2}}\right). \end{aligned} \quad (11)$$

3. The idea here was to cast the Hamiltonian of the vacuum into the form

$$H = \sum_{\mathbf{q}, \lambda} \left(\frac{|\pi_{\mathbf{q}\lambda}|^2}{2m} + \frac{m\omega_{\mathbf{q}}^2 |\phi_{\mathbf{q}, \lambda}|^2}{2} \right), \quad (12)$$

where $[\phi_{-\mathbf{q}\lambda}, \pi_{-\mathbf{q}'\lambda'}] = i\hbar\delta_{\mathbf{q}, \mathbf{q}'}\delta_{\lambda\lambda'}$ are canonical. I wanted you to work by analogy with the simple harmonic oscillator, decorating the operators with indices.

(a) Write the well-known field energy

$$H = \int d^3x \left[\frac{\epsilon_0 E^2}{2} + \frac{(\nabla \times \vec{A})^2}{2\mu_0} \right].$$

in momentum space (consider a finite system, with a discrete summation over momenta) and use this to read off m , $\omega_{\mathbf{q}}$ and $\Delta x_{\mathbf{q}\lambda}$ in terms of ϵ_0 , c and q . Here we are basically doing dimensional analysis.

We begin by identifying $\vec{\pi}(x) = -\epsilon_0 \vec{E}(x)$, so that the energy is now

$$H = \int d^3x \left[\frac{\vec{\pi}^2}{2\epsilon_0} + \frac{(\nabla \times \vec{A})^2}{2\mu_0} \right]. \quad (13)$$

Next, let us Fourier transform, remembering that $\nabla \rightarrow i\vec{q}$, to get

$$H = \sum_{\mathbf{q}, \lambda=1,2} \left[\frac{|\vec{\pi}_{\mathbf{q}\lambda}|^2}{2\epsilon_0} + \frac{|(\vec{q} \times \vec{A}_{\mathbf{q}\lambda})|^2}{2\mu_0} \right]. \quad (14)$$

where we use the modulus sign to indicate $|\vec{\pi}_{\mathbf{q}}|^2 = \vec{\pi}(\mathbf{q})^\dagger \cdot \vec{\pi}(\mathbf{q}) = \vec{\pi}(-\mathbf{q}) \cdot \vec{\pi}(\mathbf{q})$. The λ are the two polarizations, which we put in because we know that the requirement $\nabla \cdot \vec{A} = 0$ (Coulomb gauge), means that in momentum space $\vec{q} \cdot \vec{A}_{\mathbf{q}} = 0$, which means there must be two polarizations perpendicular to $\vec{q}$.

(b) Write an expression for the normal mode $A_{\mathbf{q}\lambda}$ in terms of the photon creation and annihilation operators.

First, we note that in comparison with the standard form for harmonic oscillators, $\epsilon_0 \equiv m$, while $m\omega_q^2 = q^2/\mu_0$, so that $\omega_q^2 = q^2/(\mu_0\epsilon_0) = (cq)^2$, using $c^2 = 1/(\epsilon_0\mu_0)$. From this we see that

$$\Delta A_q = \sqrt{\frac{\hbar}{m\omega_q}} \equiv \sqrt{\frac{\hbar}{\epsilon_0 c q}}, \quad \Delta \pi_q = \sqrt{\hbar m \omega_q} \equiv \sqrt{\hbar \epsilon_0 c q}, \quad (15)$$

From this we can write

$$\vec{A}_{\mathbf{q}} = \sum_{\lambda} \Delta A_q \frac{(a_{q\lambda} + a_{-q\lambda}^\dagger)}{\sqrt{2}} \hat{\mathbf{e}}_{q\lambda} \quad (16)$$

where the $\hat{\mathbf{e}}_{\vec{q}\lambda}$ are a set of two orthogonal unit polarization vectors, perpendicular to $\mathbf{q}$.

(c) Similarly

$$\vec{E}_{\mathbf{q}} = i \sum_{\lambda} \omega_q \Delta A_q \frac{(a_{q\lambda} - a_{-q\lambda}^\dagger)}{\sqrt{2}} \hat{\mathbf{e}}_{q\lambda}. \quad (17)$$

Making these substitutions in (14), the Hamiltonian becomes diagonal

$$\hat{H} = \sum_{\mathbf{q}, \lambda=1,2} \hbar \omega_q \left[\hat{n}_{\mathbf{q}\lambda} + \frac{1}{2} \right] \quad (18)$$

where $[a_{\mathbf{q}\lambda}, a_{\mathbf{q}'\lambda'}^\dagger] = \delta_{\mathbf{q}\mathbf{q}'} \delta_{\lambda\lambda'}$ and $n_{\mathbf{q}\lambda} = a_{\mathbf{q}\lambda}^\dagger a_{\mathbf{q}\lambda}$ is the number operator.

- (d) Well, we've skipped a few steps, but the results are basically correct. We haven't spoken about gauge invariance and it is not obvious what has happened to the scalar potential. But at the expense of glossing over a few details, we now understand how the photon field in the vacuum breaks up into discrete quanta.