Physics 618 Third Lecture Jan. 24, 2017

Homework 2 has been posted and is due on Feb. 2. Solutions to homework 1 will be posted Thursday night (assuming you all get your homeworks in on time). They will be accessible from the homeworks page, but only if you log on with username physics618 and the password I will tell you now.

Review

Last time we discussed some more properties of a group. We discussed **isomorphisms** between groups, and the idea that such groups are considered **equivalent** and not counted separately when we count groups. We also discussed **homomorphisms**, maps from one group into another, not necessarily onto, and defined the **kernal** of the homomorphism as the subset mapped into the identity, and observed that that is a subgroup, indeed a **normal subgroup**.

We discussed **conjugacy**, $B \sim C \quad \text{if} \quad \exists A \ni A^{-1}BA = C$. We observed that conjugacy is an **equivalence relation** and divides the group into disjoint **conjugacy classes**. We also discussed **left cosets** of a subgroup $H \subset G$ with respect to an element $g \in G$ as the the subset

$$gH := \{gh | h \in H\}$$

and similarly for right cosets Hg. These are also each an equivalence relation, dividing the group into disjoint cosets, the number of which is called the **index** of H in G, and, for finite groups,

$$\text{index of } H \text{ in } G = \text{order}(G)/\text{order}(H).$$

In particular, this says the order of a subgroup must divide the order of the group. Here the **order** of a finite group is the number of elements in the group.

If the left cosets are the same as the right cosets, H is a **normal subgroup**. Then we can also define the **quotient group** G/H whose elements are the cosets gH with composition law

$$xH \odot yH = xyH.$$

We also defined the **direct product** of two groups, the elements of which are pairs, one element from each group, and the composition law is just to treat each half according to its group. We noted that in general,

$$G \neq (G/H) \times H.$$

We briefly discussed the group S_n of **permutations** on n objects, giving two notations for the elements, one in terms of onto maps $\mathbb{Z}_n \to \mathbb{Z}_n$, and one as products of cycles. I claimed that every permutation could be written as a product of 2-cycles, or **transpositions**, in many ways, but only as a product of an even number or as a product of an odd number, depending on the permutation. Thus we divided permutations into even ones and odd ones, and observed that there is a homomorphism into $\mathbb{Z}_2$ according to even- or odd-ness. The kernal of this map, the even permutations, forms a normal subgroup called A_n, the **alternating group**. We also discussed cycles, and that any permutation could be written as a product of disjoint cycles.

Today

If anyone wants me to clarify any points I speeded over on permutations let's do that first. We will cover permutations much more thoroughly just after the midterm. Then, just to finish off the "Groups" chapter, we will list all the groups of order ≤ 7.

Then we turn to the next chapter, exploring the concept of **representations** of the group. In our first class we discussed how the states of a quantum system invariant under a symmetry group must decompose into vector subspaces for each energy eigenvalue, and the group operates as linear transformations on each subspace. If the subspace is finite-dimensional, as it will be for bound states of a system, for example, the linear transformation corresponding to each group element is represented as a finite matrix. If the dimension of the subspace is ℓ, these matrices, or more precisely the map from the group G into the set of $\ell \times \ell$ matrices, is called an ℓ-dimensional representation of the group. We will begin exploring these properties today.

On the subspace of solutions to a bound-state quantum problem with a given energy, the symmetries act as linear transformations, finite matrices. These are called **representations** of the group. They are unitary if we use orthonormal bases. It is possible that such a subspace can be divided into separate subspaces which the symmetries do not mix, in which case the representation is **reducible**, but often all the states in this subspace can be mapped into each other with the symmetries, and we have an **irreducible** representation. We will prove several theorems for finite groups.

- every representation is equivalent to a unitary one.
- (Schur's First Lemma) Any matrix that commutes with all the representatives of an irreducible representation is a multiple of $\mathbb{I}$.
- (Schur's Second Lemma) Two irreducible reps are either equivalent or there is no matrix M other than zero for which $M\Gamma^i(A) = \Gamma^j(A)M$ for all $A \in G$.

We might then get to the **Great Orthogonality Theorem** and defining **characters**, which will be very strong aids in finding the irreducible representations, but we might have to wait until next time for that.