Last week we first discussed the quantum mechanics of a degree of freedom with a symmetry, but where the lowest potential is at points which are not invariant under that symmetry. In particular, we considered a scalar field ϕ which transforms like a vector under an internal rotational symmetry SO(N), with a potential $V = -\frac{1}{2}\mu^2 \sum_j \phi_j^2 + \frac{\lambda}{4}\left(\sum_j \phi_j^2\right)^2$. Clearly the Hamiltonian is invariant under rotations in this internal space, $\phi_j \to O_{jk}\phi_k$, with $O \in$ SO(N), but if μ is real, we have a *maximum* at the symmetry point $\phi = 0$ rather than a minimum, and the ground state of such a Hamiltonian will have its wavefunction dominated by ϕ's with $|\phi| \sim \mu/\sqrt{\lambda}$, each not invariant under the group. For a quantum system with a single such degree of freedom the ground state is still symmetric, but is a superposition of states peaked about values which are not symmetric.

But if we have an extended system of infinitely many coupled degrees of freedom with a symmetry that requires all the ϕ's to transform together, the different states in the symmetric superposition have zero overlap, so no transitions are possible, and the physics is best described by perturbations around one state at the minimum of the potential. In particular, for a field theory in which gradient terms provide such a coupling, we need to describe our field in terms of excursions from a particular classical ground state $\phi(x^\mu) \approx \phi_0$, with ϕ_0 a constant at the minimum of the potential.

Last time we considered this ϕ^4 theory with SO(N) symmetry. With $\phi_0 = (0, 0, \ldots, 0, \mu/\sqrt{\lambda})$, we found, after renaming $\phi_N \to \sigma + \mu/\sqrt{\lambda}$, $\phi_j \to \eta_j$ for $j = 1, \ldots, N-1$, the Lagrangian density became

$$\begin{aligned}\mathcal{L} \;=\;& \frac{1}{2}(\partial_\nu \eta_j)^2 + \frac{1}{2}(\partial_\nu\sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\,\mu\sigma^3 - \sqrt{\lambda}\,\mu\sigma(\eta_j)^2 \\ & -\frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\sigma^2\eta_j^2 - \frac{\lambda}{4}(\eta_j^2)^2.\end{aligned}$$

This is the Lagrangian for a single scalar σ of mass $\sqrt{2}\mu$ and $N-1$ massless scalars η_j, interacting with cubic and quartic terms in the lagrangian, with couplings involving positive powers of λ. That is, in the limit $\lambda \to 0$ with fixed μ, this is a theory of free particles, $N-1$ massless ones and one with mass.

This model we just discussed is only one example, and more generally our local degrees of freedom with transform as some representation of a symmetry group $\mathcal{G}$ with Lie algebra $\mathfrak{G}$, with a classical ground state ϕ_0 which is not invariant under the whole of $\mathcal{G}$ but only under a subgroup $\mathcal{K} \subset \mathcal{G}$, with Lie algebra $\mathfrak{K}$. In our example $\mathfrak{K}$ are rotations L_{jk} with $j < k < N$. Elements of $\mathfrak{G}/\mathfrak{K}$, which are rotations $L_{j\,N}$, produce changes Δ_j which, as we saw last time, give us Goldstone bosons.

Today

Now we look into the magic that occurs when the symmetry group $\mathcal{G}$ is a gauge group, so we have massless vector particles in our Lagrangian for each direction in $\mathfrak{G}$. These particles enter the covariant derivative terms for ϕ, and so the kinetic energy term for ϕ, $(D_\mu\phi)^T(D^\mu\phi)$ has terms such as $q^2\phi_N^2 A_\mu A^\mu$ which contain a term $q^2\phi_0^2 A_\mu A^\mu$. This is not a fourth order interaction term, as it would have been without spontaneous symmetry breaking, but instead a mass term for the vector fields. Thus the massless Goldstone bosons can get eaten by the corresponding massless gauge particles, making them fat (massive). This is the Higgs mechanism, responsible for the large masses of the $W^\pm$ and Z^0 weak interaction mesons of the Salam-Weinberg model of electroweak interactions.