Last time we saw that hadrons in elementary particle physics have an approximate SU(2) symmetry called isospin, under which a proton and a neutron form a isospin 1/2 multiplet, the nucleon, and pions an isospin 1 multiplet. We saw how isospin SU(2) gave us cross sections of many different π-nucleon semi-elastic scattering in terms of only two functions, an example of the Wigner-Eckhart theorem decomposition into math versus physics questions. We also saw the beginning of the quark model and the suggestion that groups larger than SU(2) might be useful. So we switched to asking what we can say about all symmetry groups of this type, semisimple compact Lie groups and their representations.

In a finite dimensional Lie algebra, there is a maximum dimensional Abelian subalgebra, of dimension called the **rank** and usually m. Choosing such a subspace, which we call the **Cartan subalgebra**, in any representation we can diagonalize a basis of it, calling the generators H_i. Then the representation D has a basis which are eigenvectors of the $\{H_i\}$ with eigenvalues $\{\mu_i\}$, an m-dimensional vector, so the states may be written $|\mu\, D\rangle$ with

$$H_i\,|\mu\, D\rangle = \mu_i\,|\mu\, D\rangle\,.$$

The eigenvalues $\vec{\mu}$ are called the **weights** of the representation D.

We can learn a lot about possible Lie algebras by looking at the adjoint representation. For the Lie algebra itself we choose a basis including the basis of the Cartan subalgebra, $\{H_i\}$ and the remaining basis elements $\{E_{\vec{\alpha}}\}$ to correspond in the adjoint representation to eigenvectors with weights we call α_i. Thus we have

$$L_a\,|L_b\rangle = |[L_a, L_b]\rangle = i f_{ab}{}^c\,|L_c\rangle \qquad \text{and} \qquad H_i\,|E_{\vec{\alpha}}\rangle = \alpha_i\,|E_{\vec{\alpha}}\rangle\,.$$

We saw that the $E_{\vec{\alpha}}$ are complex and come in pairs, with

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}, \qquad \left[H_i, E^\dagger_{\vec{\alpha}}\right] = -\alpha_i E^\dagger_{\vec{\alpha}}.$$

Thus the generators of these other dimensions $E_{\vec{\alpha}}$ act as raising and lowering operators just as $L_\pm$ did for SU(2). We call the nonzero weights of the adjoint representation the **roots**.

Today

Today we will examine how the roots act as raising and lowering operators just as $L_\pm$ did for SU(2). For any representation, starting with any eigenvector $|\mu D\rangle$ of $\{H_i\}$, each $\vec{\alpha}$ can be used only a finite number p of times before annihilating the state, and its conjugate only a finite number q of times. We will find the **"master formula"**

$$\frac{\vec{\mu}\cdot\vec{\alpha}}{\alpha^2} = \frac{q-p}{2},$$

and applying this to the adjoint representation with weights $\vec{\alpha}$, we found the "master formula"

$$\frac{\vec{\mu}\cdot\vec{\alpha}}{\alpha^2} = \frac{q-p}{2}.$$

Applying this to the adjoint representation, we will show that any two roots satisfy $\dfrac{\vec{\alpha}\cdot\vec{\beta}}{\alpha^2} = \dfrac{n}{2}$ for $n \in \mathbb{Z}$. This gives us severe restrictions on the angles between weights, and also to the ratio of their lengths, to just a few possibilities. We will see that this constraint is extremely powerful, so that in the near future we will have found all the possible semisimple finite-dimensional compact Lie groups. But before doing so we will examine how this works in the next-simplest algebra, SU(3).

Reminders:

There will be a midterm exam on Tuesday, March 7.

I currently think this will cover the material from the beginning through Dynkin diagrams (Chapter 8), but we might modify that as we get closer to the exam.

We should discuss what you can bring and use on the exam.

Homework 5 is due Thursday, February 23.

Homework 6 will probably be due Thursday, March 2. It has not yet been posted, but you should look, as it will hopefully be there by Thursday.