Chapter 16

More SSB, then Higgs

16.1 An example: SSB for SO(N)

Consider a quantum field theory with a multiplet of N *real scalar fields* with Lagrangian density

$$\mathcal{L} = \sum_{j=1}^{N} (\partial_\nu \phi_j)(\partial^\nu \phi_j) + \frac{1}{2}\mu^2 \sum_{j=1}^{N} \phi_j^2 - \frac{\lambda}{4}\left(\sum_{j=1}^{N} \phi_j^2\right)^2 .$$

This lagrangian is, of course, invariant under SO(N), rotations of ϕ in the N-dimensional internal space. $\phi_j(x) \to O_{jk}\phi_k(x)$, with O an orthogonal matrix, which leaves $\sum \phi^2$ invariant. Notice that the mass term has the wrong sign. Had that sign been minus, with a + in the potential energy, the potential's absolute minimum would have been at $\vec{\phi} = 0$, the classical vacuum would be at $\vec{\phi} = 0$, and the rotational symmetry would be intact. But with the plus in the lagrangian, $\vec{\phi} = 0$ is a local maximum of the potential energy, and not a classical ground state. Instead a classical lowest energy state will have $\vec{\phi} = \vec{\phi}_0$ with $\phi_0 := |\vec{\phi}_0| = \mu/\sqrt{\lambda}$. To get a state of lowest energy, we need not only that $V(\phi)$ is minimized at each point x, but also that the kinetic energy $(\vec{\nabla}\phi)^2$ is minimized, which means vanishing. So although the potential only tells us that $\vec{\phi}(x)$ should lie on a sphere of radius ϕ_0, minimizing the energy means it has to be the same $\vec{\phi}$ throughout space.

If we apply a global SO(N) transformation, we get a new state which is of the same energy as our state $\vec{\phi}_0$, but which has an overlap with $\vec{\phi}_0(x)$ at every point which is less that one, so that raised to the infinite power from the infinite number of points $\vec{x}$, there is zero overlap! A system in one of these ground states can never get to another, equivalent, ground state.

In all our field-theoretic considerations so far, we have assumed the vacuum arises somehow from the classical state where the fields are all zero. We know there are vacuum fluctuations, but they are fluctuations about the $\phi = 0$ state. But if there is a classical state of lower energy (or, more accurately, energy density), we should expect our vacuum and our low excitations from it to be based on this lowest-energy state ϕ_0, not the $\phi = 0$ state.

How to proceed? We can rewrite our fields $\vec{\phi}(x) = \vec{\phi}_0 + \vec{\eta}(x)$. We can choose our $\vec{\phi}_0$ to be anywhere on the minimal surface, so let us choose it in the N direction, $\phi_0 = (0, \cdots, 0, \mu/\sqrt{\lambda})$. As $\vec{\phi}_0$ is a constant, the kinetic energy term in $\mathcal{L}$ is just $\sum_{j=1}^{N}(\partial_\nu \eta_j)(\partial^\nu \eta_j)$. The potential energy as a function of η is now

$$V(\eta) = -\frac{\mu^2}{2}\left(\sum_{j=1}^{N-1}\eta_j^2 + (\eta_N + \mu/\sqrt{\lambda})^2\right) + \frac{\lambda}{4}\left(\sum_{j=1}^{N-1}\eta_j^2 + (\eta_N + \mu/\sqrt{\lambda})^2\right)^2 .$$

As η_N is now being treated differently, let's call it σ. We have

$$\begin{aligned} V(\eta) &= -\frac{\mu^4}{2\lambda} - \frac{\mu^3}{\sqrt{\lambda}}\sigma - \frac{\mu^2}{2}\sigma^2 - \frac{\mu^2}{2}\eta_j^2 \;+\; \frac{\lambda}{4}\left(\eta_j^2 + \sigma^2 + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \frac{\mu^2}{\lambda}\right)^2 \\ &= -\frac{\mu^4}{4\lambda} + \left(-\frac{\mu^3}{\sqrt{\lambda}} + \frac{\mu^3}{\sqrt{\lambda}}\right)\sigma + \left(-\frac{1}{2} + 1 + \frac{1}{2}\right)\mu^2\sigma^2 + (-1+1)\frac{\mu^2}{2}\eta_j^2 \\ &\qquad + \mu\sqrt{\lambda}\,\sigma^3 + \mu\sqrt{\lambda}\,\sigma\eta_j^2 + \frac{\lambda}{4}\left(\eta_j^2\right)^2 + \frac{\lambda}{4}\sigma^4 + \frac{\lambda}{2}\sigma^2\eta_j^2 \end{aligned}$$

where the sums η_j now have $j = 1..N-1$, with $\eta_j^2 := \sum_{j=1}^{N-1}\eta_j^2$.

Notice the linear term in σ vanishes, as it must, because the minimum of V is at $\sigma = 0$, $\eta_j = 0$. Notice also that the η_j has lost its quadratic term, so these $N-1$ degrees of freedom have become massless. Finally, notice that the σ has developed a mass $\sqrt{2}\,\mu$ with the correct positive sign in the potential. Our lagrangian has now become

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\nu\eta_j)^2 + \frac{1}{2}(\partial_\nu\sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\,\mu\sigma^3 - \sqrt{\lambda}\,\mu\sigma(\eta_j)^2 \\ &\qquad -\frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\sigma^2\eta_j^2 - \frac{\lambda}{4}(\eta_j^2)^2. \end{aligned}$$

We have quartic terms for all the fields, with the correct signs to keep energy bounded from below, though we now have cubic interactions, of the σ with the η's and with itself. The theory still has a symmetry under rotations in the $N-1$ dimensional space $j = 1..N-1$, but it has lost symmetry under rotations which include the N'th dimension. It also does not have symmetry under $\sigma \leftrightarrow -\sigma$.

The model we have just considered is called the linear sigma model. With $N = 4$, we are left with 3 massless fields. This was taken in the '60's as a model describing the isotriplet of pi mesons, which are light compared to all other hadrons. The pions are made of the (u, d) isodoublet ψ of quarks and their antiquarks, we would have an isospin SU(2) symmetry with charges $\frac{1}{2}\int \psi^\dagger \vec{\tau}\psi$, but if the quarks are massless, there is also a chiral SU(2) symmetry with additional charges $\frac{1}{2}\int \psi^\dagger\gamma_5\vec{\tau}\psi$, together forming an SU(2)×SU(2) ≡ SO(4) symmetry group. If we imagine that this symmetry is somehow spontaneously broken, and in addition there is a small explicit breaking due to small quark masses, it might explain both the near masslessness of the pions and also the correction to that, connected to the not-quite-conservation of the axial vector current in weak interactions.

16.2 The Higgs Mechanism

Last time we discussed spontaneous symmetry breaking with a more general symmetry, a theory with a multiplet of scalar fields ϕ transforming as a unitary irreducible representation of a symmetry group $\mathcal{G}$. The Lagrangian was invariant under $\mathcal{G}$, but possibly the lowest energy state (vacuum), about which we do perturbation theory, was not invariant. The vacuum expectation value ϕ_0 may be left invariant under a subgroup $\mathcal{K} \subset \mathcal{G}$, and the dimensions in the Lie Algebra $\mathfrak{G}$ of $\mathcal{G}$ which do not leave ϕ_0 invariant ($\mathfrak{G}/\mathfrak{K}$, generators of the cosets of $\mathcal{G}/\mathcal{K}$) give rise to massless scalar particles known as Goldstone Bosons.

But in chapters 13-14, we developed the magnificant gauge theories with *local* symmetry, where the matter fields ϕ can be transformed independently at each space-time point, but at the expense of adding gauge fields $\mathcal{A}_\mu$ taking values in $\mathfrak{G}$ to the theory. These fields entered the lagrangian both by modifying the gradient operator into a covariant derivative, but also with a self-interaction $-\frac{1}{4}\sum_d F^{(d)}_{\mu\nu}F^{(d)\,\mu\nu}$. We saw that these particles described *massless* vector particles, each having only two physical degrees of freedom despite having four components μ. So if we combine these two ideas, we seem to be building up a theory with both massless vectors and massless scalars.

But things are more interesting than that.

16.2.1 The Abelian Higgs Model

Our first example will be a simple U(1) theory with a complex scalar ϕ. As with electromagnetism, this means we have one gauge field A_ν and a local phase symmetry $\phi \to e^{-i\alpha(x)}\phi$, $A_\nu \to A_\nu + \frac{1}{q}\partial_\nu \alpha(x)$. We take the lagrangian density to be

$$\mathcal{L} = (D^\nu\phi)^\dagger D_\nu\phi - \frac{1}{4}F_{\nu\rho}F^{\nu\rho} + \mu^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2. \tag{16.1}$$

Were it not for the mass term having the wrong sign, this would be simply a theory of a charged scalar interacting with photons. There would be the two real scalar degrees of freedom from ϕ, and the two transverse polarizations of the photon, for a total of four. But note that the mass term does have the wrong sign, so we will have spontaneous symmetry breaking. The classical minimum of the potential has $\phi^\dagger\phi = \frac{2\mu^2}{\lambda}$, so we choose our vacuum to be at $\phi = v/\sqrt{2}$ where v is real[1] and $v = 2\mu/\sqrt{\lambda} > 0$. We need to reexpress the two ϕ degrees of freedom. Rather than doing this with cartesian coordinates, let's write ϕ in terms of a radial coordinate coordinate $|\phi| = v + h(x)$ and an angular coordinate or phase angle θ/v. Thus

$$\sqrt{2}\phi(x) = (v + h(x))\, e^{-i\theta(x)/v}.$$

When the classical vacuum state has all fields at zero, the particle content is found by looking at quadratic terms in the Lagrangian, which give linear wave equations for free particles with masses determined by the term quadratic in the fields without derivatives. Thus the terms in $(D^\nu\phi)^\dagger D_\nu\phi$ which involve the gauge field A_μ as well as some ϕ's give interactions but don't affect the "free particle" masses of ϕ or A. But if the symmetry is broken and the low-lying states are expanded about $\phi_0 = v/\sqrt{2}$, the covariant term $[(\partial^\nu - iqA^\mu)\phi^\dagger][(\partial_\mu + iqA_\mu)\phi$ includes a term $q^2A^\mu A_\mu\phi^\dagger\phi \sim \frac{1}{2}q^2v^2A^\mu A_\mu$ which is a mass term, giving the photon a mass qv.

Things are actually a bit more complicated than this, because gauge invariance involves taking the ϕ field away from our chosen vacuum state. To actually understand the particle content, or physical degrees of freedom, we need to recall that choosing a gauge condition removes some of these. Given

[1] The funny $\sqrt{2}$ factors are because we define $\sqrt{2}\,\phi = \phi_1 - i\phi_2$, so $\phi_{j,0} = (v, 0)$.

an arbitrary initial A_μ and complex ϕ, we could have chosen the gauge to make ϕ real everywhere, which would eliminate the Goldstone boson component. This is called the unitary gauge. But using our gauge freedom this way means we don't have it available for the Lorenz gauge choice or the Feynman-'t Hooft gauge, and the Ward identity then does not make the longitudinal and time components of the A_μ field vanish. Thus we see, in the unitary gauge, that the photon has eaten the Goldstone boson and become massive.

16.2.2 Broken Non-Abelian Gauge Theory

Let us consider the simplest non-Abelian example, with the group $\mathcal{G}$ =SU(2). This has a three-dimensional Lie algebra and so we begin with three gauge particles A^a_μ. Our scalar particles will need to transform under some representation of SU(2). First let's consider the isospin 1/2 representation, a complex doublet ϕ. The algebra is represented by $L_a = \tau_a/2$, so the covariant derivative is

$$D_\mu\phi = (\partial_\mu - iqA^a_\mu\tau_a/2)\phi.$$

With the ϕ in (16.1) now refering to this complex doublet, we still see the minimum of the potential requires $2\phi^\dagger\phi = v$, but now we choose ϕ_0 not only to be real but to have zero upper component,

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}.$$

Then the $|D_\mu\phi|^2$ term quadratic in A is

$$\frac{g^2}{8}(0,v)\tau_a\tau_b\begin{pmatrix}0\\ v\end{pmatrix}A^a_\mu A^{b\,\mu} = \frac{g^2v^2}{8}A^a_\mu A^{a\,\mu},$$

where the symmetry of $A^a_\mu A^{b\,\mu}$ under $a \leftrightarrow b$ enabled us to replace $\tau_a\tau_b$ by $\frac{1}{2}\{\tau_a,\tau_b\} = \delta_{ab}$. Thus each of the three gauge particles develops a mass $m_A = gv/2$, and there are no massless vectors left. We will have one remaining massive real scalar left, coming from the oscillations around $v/\sqrt{2}$ of the real part of the lower component of the doublet. The other degrees of freedom have been eaten, fixed to be zero by choice of gauge (called unitary gauge).

Now we might have chosen ϕ to transform differently. For example, we might have chosen an isospin 1 real field, with *three real components.* This transforms under the adjoint representation of SU(2), so now

$$(D_\mu\phi)_a = \partial_\mu\phi_a + g\epsilon_{abc}A^b_\mu\phi_c,$$

and the A^2 term in the lagrangian comes from $\frac{1}{2}(D_\mu\phi)^2 = \frac{g^2}{2}(\epsilon_{abc}A^b_\mu\phi_{0\,c})^2$. The theory has spherical symmetry in isospace, so if the $V(\phi)$ term takes its minimum value for $|\phi| = v \neq 0$, it can be anywhere on a sphere of radius v, and we can choose that to be along the third axis, so $\phi_0 = (0, 0, v)$, and $(\epsilon_{abc}A^b_\mu\phi_{0\,c})^2 = v^2\epsilon_{ab3}A^b_\mu\epsilon_{ac3}A^{c\,\mu} = v^2[(A^1_\mu)^2 + (A^2_\mu)^2]$. Thus two of the gauge particles develop masses, $m_1 = m_2 = gv$, but the third remains massless. And ϕ_3 becomes the sole surviving massive real scalar.

In both of these examples, any point that could be the minimum of $V(\phi)$ was equivalent to any other under the symmetry, but that is not always the case. One example[2] is an SU(3) gauge group with an octet scalar. If ϕ_0 is in the λ_8 direction, the SU(3) is broken into SU(2)×U(1), so those four gauge particles remain massless, while the other four develop equal positive masses. But if ϕ_0 is in the λ_3 direction, only A^3 and A^8 remain massless, the symmetry is broken to U(1)×U(1), and four of the other vector particles develop a mass M and the other two a mass $2M$.

16.2.3 A Side Comment on g

When the Killing form is positive definite, as it is for the semisimple groups we are considering, it presents a natural way to normalize the basis vectors of the Lie algebra. This gives a natural metric in group space, and such groups are compact sets, so they have a natural size. But the symmetries act linearly on the scalar or spinor fields, so there is no natural strength by which a gauge field should act on a matter field, so we have a parameter, a kind of charge, g, which we have always seen in our covariant derivatives of matter fields. The strengths by which the different gauge fields act on the matter fields is, however, determined by the matter-field representation, if it is irreducible.

If, however, the gauge group is a direct product of two groups, the covariant derivative will be a sum over gauge fields from the two different groups, and the strength with which each couples will not be constrained. So there will be separate coupling constants for the two components.

[2]Peskin and Schroeder, pp. 696-697.

16.2.4 SU(2)×U(1) Gauge Theory with Isodoublet Higgs

Now let us consider the group which will give us the Glashow-Salam-Weinberg model of the electroweak interactions, which is a major component of the standard model. The group is SU(2)×U(1). The gauge particles are three $\vec{W}_\mu$'s for SU(2) and one B_μ for U(1). The field strength for the W's will be called F

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu - g\vec{W}_\mu \times \vec{W}_\nu,$$

and that for B will be called $B_{\mu\nu}$,

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

All of the particle fields we know about, quarks and leptons, will be included, but we concentrate for now on a doublet of complex scalars ϕ which will spontaneously break the symmetry. Acting on ϕ, the covariant derivative is

$$D_\mu = \partial_\mu + i\frac{g}{2}\vec{\sigma}\cdot\vec{W}_\mu + i\frac{g'}{2}B_\mu.$$

We are looking to have vector fields with charge, so the doublet needs to have different charges for its two components, and we want the one that develops a vacuum expectation value to be neutral, so we write $\phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$ and $\phi_0 = \begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix}$. The general field configuration for ϕ can now be written as

$$\sqrt{2}\,\phi = e^{-i\vec{\theta}\cdot\vec{\sigma}/2v}\begin{pmatrix}0\\ v+h(x)\end{pmatrix},$$

but we will immediately go to the unitary gauge, which undoes the exponential factor, leaving from ϕ only the real scalar Higgs field $h(x)$ with a mass $\sqrt{2}\,\mu = v\sqrt{\lambda/2}$. But the $|D_\mu\phi|^2$ term now gives us a term

$$\frac{1}{8}(0,v)[g\vec{\sigma}\cdot\vec{W}_\mu + g'B_\mu]^2\begin{pmatrix}0\\ v\end{pmatrix} = \frac{v^2}{8}\left[g^2(W_1^2+W_2^2) + (gW_3 - g'B)^2\right].$$

We see that $m_{W_1} = m_{W_2} = gv/2$, but the mass matrix is not diagonalized by our choice of basis vectors for the other two gauge fields, and we need to choose a new basis by rotating in the $W_3{-}B$ plane,

$$\begin{aligned} Z^\mu &= \cos\theta_W W_3^\mu - \sin\theta_W B^\mu \\ A^\mu &= \sin\theta_W W_3^\mu + \cos\theta_W B^\mu \end{aligned}$$

where

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}.$$

Then the gauge field mass terms are $\frac{1}{2}M_W\left(W_1^\mu W_{1\,\mu} + W_2^\mu W_{2\,\mu}\right) + \frac{1}{2}M_Z Z^\mu Z_\mu$, with $M_W = gv/2$, $M_Z = \frac{1}{2}v\sqrt{g^2+g'^2} = M_W/\cos\theta_W$. We see that the A field does not pick up any mass, so we identify it as the photon field.

Now look at the covariant derivative in terms of the new basis:

$$\begin{aligned}
D_\mu\phi &= \partial_\mu\phi + \Big[i\frac{g}{2}\sigma_+ W^- + i\frac{g}{2}\sigma_- W^+ \\
&\quad + i\frac{g}{2}(\sigma_3(\cos\theta_W Z + \sin\theta_W A) \\
&\quad + i\frac{g}{2}\frac{\sin\theta_W}{\cos\theta_W}(\cos\theta_W A - \sin\theta_W Z))\Big]\phi \\
&= \partial_\mu\phi + \Big[i\frac{g}{2}\sigma_+ W_\mu^- + i\frac{g}{2}\sigma_- W_\mu^+ + ig\sin\theta_W\frac{1+\sigma_3}{2}A_\mu \\
&\quad + i\frac{g}{2\cos\theta_W}(\sigma_3 - (1+\sigma_3)\sin^2\theta_W)Z_\mu\Big]\phi
\end{aligned}$$

We see that the photon field A_μ couples only to the upper component of ϕ, and with a charge $e = g\sin\theta_W$, which is therefore the unit of electromagnetic charge e, that of a positron.

This constitutes the bosonic part of the standard electroweak theory of the standard model, but so far we haven't introduced any of the particles that have electroweak interactions! So now it is time for us to introduce the leptons and quarks into this model.

16.3 Adding Leptons and Quarks

In the last section we described the gauge group for the electroweak interactions as SU(2)×U(1), broken by the complex Higgs doublet. We saw that the neutral gauge particles W^0 and B mix, leaving an unbroken gauge symmetry for electromagnetism and a neutral Higgs.

Now we will introduce the quarks and leptons, so we can actually describe electromagnetic and weak interactions.

Fortunately, the doublet nature of weak interactions was apparent long before the sophisticated nature of our model was known. The weak current, being charged, connected pairs of particles with charges differing by one unit, and in fact looked like a collection of doublets, whether proton-neutron or $\nu_e - e^-$ or $\nu_\mu - \mu^-$ or $u{-}d$ quarks. The first and last, of course, are the origins of hadronic isospin, and are only approximately the doublets we are going to want for weak isospin. It has been known since 1956 that the weak interactions are not parity invariant. All of the fundamental matter particles we will want to introduce will be spinors, but the left handed components will behave differently from the right handed ones under weak interactions. Their left-handed pieces will be doublets under the weak SU(2) gauge, while their right-handed pieces will be singlets. These spinors will be massless in the fundamental lagrangian, as we can't have a gauge-invariant mass term coupling a doublet and a singlet. There will be three generations of leptons and three generations of quarks. The doublets all interact with the $\vec{W}_\mu$ with the strength $g\vec{\sigma}/2$, while the singlets, of course, do not. The upper component of each doublet has a charge one unit higher than the lower component, so $\Delta Q = \Delta t_3$, where t and t_3 are the weak isospin and its third component. All multiplets interact with B_μ with a strength $ig'y/2$, where the *weak hypercharge* y varies from multiplet to multiplet, -1 for the left-handed leptons, -2 for the right-handed negatively charged leptons, 0 for the right-handed neutrinos (if they exist). The left-handed quarks have $y = 1/3$, while the right-handed quarks, in order to have the same charge as their left-handed components, have $y = 4/3$ for u, c, and t, and $y = -2/3$ for d, s, and b. Note in all cases the charge Q (in units of $e > 0$) is

$$Q = t_3 + \frac{1}{2}\,y.$$

Thus each of the spinor fields enters the lagrangian with

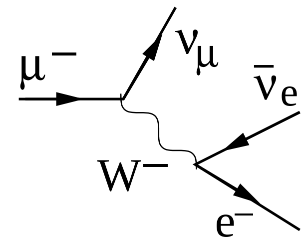
$$\mathcal{L} = i\bar{\psi}\not{D}_L \frac{1-\gamma_5}{2}\psi + i\bar{\psi}\not{D}_R \frac{1+\gamma_5}{2}\psi$$

with

$$\begin{aligned} D_{L\,\mu} &= \partial_\mu + ig\vec{\sigma}\cdot\vec{W}_\mu/2 + ig'y\mathbb{1}B_\mu/2, \\ D_{R\,\mu} &= \partial_\mu + ig'yB_\mu/2. \end{aligned}$$

though for the right-handed neutrinos, as their $t = y = 0$, there is no way for them to interact and no way for us to know, in the standard model, whether or not they exist!

Let us consider the decay of the muon, $\mu^- \to e^- + \nu_\mu + \bar\nu_e$, This proceeds by the second order interaction with W exchange. We note that $\vec\sigma \cdot \vec W_\nu = \frac{1}{\sqrt2}(\sigma_+ W_\nu + \sigma_- W_\nu^\dagger)$ where $\sigma_\pm = \frac12\sigma_1 \pm \sigma_2$ and $W^\nu = (W_1^\nu - iW_2^\nu)/\sqrt2$. The reason for this notation is that $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and W has the right weight for a propagator. Now the relevant part for the electron field is σ_- and for the muon σ_+, so the invariant amplitude is



$$i\mathcal{M} = \left(i\frac{g}{\sqrt2}\bar\psi_e\gamma^\rho(1-\gamma_5)\psi_{\nu_e}\right)\frac{i(-g_{\rho\tau}+k_\rho k_\tau/M_W^2)}{k^2-M_W^2}\left(i\frac{g}{\sqrt2}\bar\psi_{\nu_\mu}\gamma^\tau(1-\gamma_5)\psi_\mu\right).$$

But the muon only has a mass of 105 MeV so the momentum transfer $k \ll M_W = 80.38$ GeV, so we can pretty well put k to zero, and

$$i\mathcal{M} = i\frac{g^2}{2M_W^2}\bar\psi_e\gamma^\rho(1-\gamma_5)\psi_{\nu_e}\bar\psi_{\nu_\mu}\gamma_\rho(1-\gamma_5)\psi_\mu$$

which is exactly the old Fermi four-fermion interaction with coupling constant

$$\frac{G_F}{\sqrt2} = \frac{g^2}{8M_W^2} = 1/2v^2.$$

From the long-measured value $G_F = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$ we find $v \approx 246$ GeV.

The same diagram, charge conjugated, will give the scattering cross section for $\nu_\mu + e^- \to \nu_e + \mu^-$, which might be used to detect muon neutrinos. But it was also discovered in bubble chambers in 1973 that there was also an elastic cross section, $\nu_\mu + e^- \to \nu_\mu + e^-$, which must proceed by a neutral current, the Z. From the effective coupling constant here, together with the electric charge $= g\sin\theta_W$, we have the three experimental values we need to determine g, g' and v, and thus

$$\sin^2\theta_W = 0.231, \qquad M_Z = M_W/\cos\theta_W = 91.2\,\mathrm{GeV}.$$

16.3.1 Quark Weak Interactions

As we mentioned, all the fundamental spinors have their left-handed components as part of a weak isodoublet, and their right handed components immune to the $\vec{W}$ gauges, feeling only the B in their covariant derivatives. Thus the left handed components are

$$\left\| \begin{array}{cccc|ccc} & \begin{pmatrix}\nu_e\\ e^-\end{pmatrix} & \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix} & \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix} & \begin{pmatrix}u\\ d'\end{pmatrix} & \begin{pmatrix}c\\ s'\end{pmatrix} & \begin{pmatrix}t\\ b'\end{pmatrix} \\ y= & -1 & -1 & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right\|$$

all with $t = 1/2$. The right handed quarks and leptons do not come in doublets. The leptons e^-_R, μ^-_R, τ^-_R have $y = -2$, the quarks whose flavor cousins are $t_3 = +1/2$ (u_R, c_R, t_R) have $y = 4/3$ and the others $y = -2/3$ so as to have total electromagnetic charges of $2/3$ and $-1/3$ respectively.

We have distinguished between hadronic isospin and weak isospin, but we haven't discussed what that difference is. When we classify the u and d as a hadronic isospin doublet, we are basing the assignment of what particle comes from $T_- |u\rangle$ on the overwhelming strength of the strong interaction. But what about $t_- |u\rangle$, the action of the weak isospin? It gives us a quark of charge $-1/3$, but might it not be a mixture of the d, s and b quarks as distinguished by strong interactions and their very different masses? We have written the weak isodoublets in terms of primed quark fields to distinguish them from the strong interaction conserved flavors. We did not bother with the upper components, because we can choose the weak upper components to be whatever the strong interactions wanted by definition of which generation is which. But then the lower components are what the W's produce from the upper ones. These three fields, d', s' and b' must be a unitary transformation of the mass eigenstates d,s and b, so we may write

$$\begin{pmatrix}d'\\ s'\\ b'\end{pmatrix} = \mathbf{V} \begin{pmatrix}d\\ s\\ b\end{pmatrix} \qquad \text{with} \quad \mathbf{V} = \begin{pmatrix}V_{ud} & V_{us} & V_{ub}\\ V_{cd} & V_{cs} & V_{cb}\\ V_{td} & V_{ts} & V_{tb}\end{pmatrix}$$

where $\mathbf{V}$ is the CKM or Cabbibo, Kobayashi, Maskawa matrix.

Note that if there were no such discrepancy between weak and strong isospin, that is, if $\mathbf{V}$ were $= 1\!\!1$, there would be conservation of (strangeness + charm), and K mesons could not decay. So part of the CKM matrix, V_{us}, was proposed by Cabbibo in 1963 before three generations or even the charmed quark were known.

For the leptons, as far as we know[3], each generation does have its own conservation law, so there is no mixing, and so only the leptonic equivalent of V_{ud}, V_{cs}, V_{tb} are nonzero, and are pure phase transformations. Also, because the phase of each field is arbitrary, we can adjust phases to make the leptonic equivalent of $\mathbf{V}$ be the identity. For the quarks, we can adjust the six phases of the weak and strong fields, but adjusting them all by the same amount leaves $\mathbf{V}$ unchanged, so there are 5 adjustable parameters of the nine real parameters that a 3×3 unitary matrix has. Thus there are four parameters which need to be measured to know $\mathbf{V}$.

One important consequence of this possible mixing is that the $\mathbf{V}$ matrix cannot be made real by the choice of field phases. If it could, it would be an orthogonal 3×3 matrix with only three adjustable parameters. That there is room in $\mathbf{V}$ for a complex parameter which cannot be made real means that CP conservation, which is equivalent to T conservation in QFT, can be violated, as time reversal involves complex conjugation. If there were only two generations of quarks, this would not be the case, and the weak interactions would not give an explanation of CP violation, such as seen in the $K^0 - \bar{K}^0$ system and also the corresponding D^0 and B^0 systems.

16.3.2 Fermion Masses

We have seen that the left-handed and right-handed parts of the spinors transform differently under weak isospin, so a simple mass term $m\bar{\psi}\psi$ is not invariant. This is why the theory we are using, before symmetry breaking, has no massive spinors.

But the vacuum expectation value ϕ_0, which breaks the symmetry, provides a mechanism for generating mass. Under weak isospin, $\bar{\psi}_L$ transforms properly[4] so that $\bar{\psi}_L\phi$ is a scalar under weak isospin. So

$$\mathcal{L}^{(e)}_{\text{Yuk}} = -g_e\bar{\psi}_{e,L}\phi\psi_{e,R} + h.c.$$

obeys all required symmetries, and reduces to $-\dfrac{g_e v}{\sqrt{2}}\left(\bar{e}_L e_R + \bar{e}_R e_L\right) = -m_e\bar{\psi}_e\psi_e$,

[3] Wrong! Wrong! Wrong! We now have conclusive evidence for neutrino mixing, which means one flavor of neutrino can turn into a different one, so there is no conservation of the number of leptons of each individual flavor. It is still not known if there is conservation of the total number, $N_e + N_\mu + N_\tau$.

[4] The convoluted language, rather than saying $\bar{\psi}_L$ is $t = 1/2$, is because the conjugate to a standard $t = 1/2$ representation is *equivalent* to the standard representation, it is not equal to it.

when we discard terms cubic in the fluctuating fields, replacing ϕ by ϕ_0, This provides the electron with a mass $m_e = g_e v/\sqrt{2}$. Thus we see that the vacuum breakdown provides mass to the fermions as well as the $W^\pm$, provided there is a Yukawa coupling in the Lagrangian of the electron with the Higgs field. But this means there is also a coupling to the fluctuating part of the Higgs field, $-\frac{g_e v}{\sqrt{2}}\hat{h}(x)\left(\bar{e}_L(x)e_R(x)+\bar{e}_R(x)e_L(x)\right) = -\frac{gm_e}{2M_W}\hat{h}(x)\bar{e}(x)e(x)$.

The other bottom components of the isodoublets can develop masses in the same way as the electron. But what about the upper components? The secret comes from closer examination of the transformation of the standard isospin 1/2 representation, under which

$$\psi_a \to \left(e^{i\vec{\alpha}\cdot\vec{\sigma}/2}\right)_{ab}\psi_b.$$

The conjugate representation has

$$\psi_a^\dagger \to \left(e^{-i\vec{\alpha}\cdot\vec{\sigma}^T/2}\right)_{ab}\psi_b^\dagger.$$

Thus it does not transform like the standard representation, but $\chi_a := \epsilon_{ac}\psi_c^\dagger$ does, where $\epsilon_{ab} = (i\sigma_2)_{ab}$ is the two-dimensional antisymmetric Levi-Civita tensor $\epsilon_{12} = 1 = -\epsilon_{21}$, $\epsilon_{11} = \epsilon_{22} = 0$ in isospin space. To see this,

$$\chi_a \;:=\; \epsilon_{ac}\psi_c^\dagger \to \left[(i\sigma_2)\left(e^{-i\vec{\alpha}\cdot\vec{\sigma}^T/2}\right)\psi^\dagger\right]_a = \left[\left(e^{i\vec{\alpha}\cdot\vec{\sigma}/2}\right)(i\sigma_2)\,\psi^\dagger\right]_a = \left(e^{i\vec{\alpha}\cdot\vec{\sigma}/2}\chi\right)_a.$$

So $\epsilon_{ab}\bar{\psi}_{u,L\,b}$ transforms like ϕ_a and $\phi_a^\dagger\epsilon_{ab}\bar{\psi}_{u,L\,b}$ is invariant, and can be contracted with $\psi_{u,R}$. Of course $\phi^\dagger_{0\,a}\epsilon_{ab} = (\frac{v}{\sqrt{2}}, 0)$. Thus the $u-d$ quark Yukawa interaction is $\mathcal{L} = -\lambda_d\bar{\psi}_{u,L}\cdot\phi\, d_R - \lambda_u\bar{\psi}_{u,L}(i\sigma_2)\phi^\dagger u_R + h.c.$

Thus we see that we can introduce any spinor masses to the quarks and the charged leptons we wish, so our theory can accommodate any quark and lepton masses, but does not predict them. In so doing, the couplings of these fermions to the higgs field is determined, with a strength proportional to the induced fermion mass. Thus you might think the experimentalists found the Higgs by looking for $b\bar{b}$ and their decays, but unfortunately that channel has so much background this is impossible to extract. It was actually the coupling to the gauge particles, photons and Z's, together with $\tau\bar{\tau}$, that were detected.