Physics 618 Homework #6

Due: Thursday, March 2, 2017 at 4:00 PM

Reminder: There will be a midterm exam on Tuesday, March 7. You will be allowed to use two pages ($8\frac{1}{2} \times 11$ inch) of handwritten notes.

1 [10 pts] Consider a semisimple Lie algebra $\mathcal{L}$ of rank 2, with simple roots $\vec{\alpha}$ and $\vec{\beta}$. Let $p = -2\vec{\alpha} \cdot \vec{\beta}/\alpha^2$, and suppose $|\vec{\alpha}| \leq |\vec{\beta}|$.
a) Show that if $p = 0$, the algebra is a direct sum of two algebras which are each an $SU(2)$. Thus the group is $SU(2) \times SU(2)$.
b) For $p = 1$, find which positive linear combinations of the simple roots are roots. Give a diagram of the root vectors. Check to make sure you have included all roots.
c) Do the same thing for $p = 2$.
d) For case (b), with $p = 1$, work out the entire algebra, including normalizations. Choose your H_1 and H_2 so that one root points along the H_1 axis.

2 [5 pts] [Note: this is Georgi's problem 5.B. Also note that $[A, B]_+$ is the anticommutator, $[A, B]_- = [A, B]$ is the commutator.]
Suppose X_a are $N \times N$ matrices satisfying

$$[X_a, X_b] = i f_{abc} X_c$$

and $b_i^\dagger$ and b_i are creation and annihilation operators satisfying ($i = 1$ to N)

$$\begin{aligned} \left[b_i, b_j^\dagger\right]_\pm &= \delta_{ij} \\ \left[b_i^\dagger, b_j^\dagger\right]_\pm &= [b_i, b_j]_\pm = 0. \end{aligned}$$

[That is, these relations hold either for commutators or for anticommutators, but not both simultaneously. Show the following holds in either case.]

Show that the operators

$$\chi_a = \sum_{i,j} b_i^\dagger \, [X_a]_{ij} \, b_j$$

satisfy

$$[\chi_a, \chi_b] = i f_{abc} \chi_c$$

3 [10 pts] [Note: this is Georgi's problem 6.C] Consider the simple Lie algebra formed by the 10 matrices σ_a, $\sigma_a \tau_1$, $\sigma_a \tau_3$, and τ_2 where σ_a and τ_a are Pauli matrices in orthogonal spaces. That is, we are considering a *subspace* of the direct product space $GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$ where the σ's act on the first space and the τ's on the second. You might want to look at Georgi problem 3.E. Take $H_1 = \sigma_3$ and $H_2 = \sigma_3 \tau_3$ as the Cartan subalgebra. Find

(a) the roots of the adjoint representation and

(b) the weights of the four dimensional representation generated by these matrices.