Physics 618 Homework #5

Due: Thursday, Feb. 23, 2017 at 4:00 PM

1 [10 pts] $SU(1,1)$ is the set of 2×2 complex matrices of determinant 1 which leave invariant the form $|z_1|^2 - |z_2|^2$.
a) Find all elements in an infinitesimal neighborhood of $\mathbb{1}$, and parameterize them.
b) What is the dimension of this group? Give a well-chosen basis for the Lie algebra.
c) Find the structure constants and the Killing form. Is the group semisimple?
d) Evaluate $e^{i\Sigma \nu_j L_j}$ for finite ν_j, at least well enough to discuss the range of the parameters. Is the group compact? Is it simply connected?

2 [5 pts] Show that there does not exist a two-dimensional semisimple Lie algebra.

3 [10 pts] Let $R(\vec{\omega})$ be a rotation $\vec{r} \to \vec{r}\,'$ where, to first order in the angle of rotation ω, $\vec{r}\,' = \vec{r} - \vec{\omega}\times\vec{r}$. R acts on the space of functions on $\mathbb{R}^3$ by $Rf = f'$, with $f'(\vec{r}\,') = f(\vec{r})$, and if for infinitesimal $\vec{\omega}$ we write $R = 1 + i\vec{\omega}\cdot\vec{L}$,
a) show that $\vec{L} = -i\vec{r}\times\vec{\nabla}$.
b) Express this in component form, using ϵ_{ijk}, and verify that the operators L_i satisfy the $SU(2)$ Lie algebra commutation relations.
c) Show that, in spherical coordinates,

$$L_z = -i\frac{\partial}{\partial\phi}, \qquad L_\pm = \frac{1}{\sqrt{2}}\, e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right).$$

d) Show that if a set of functions $Y_{\ell m}$ transform as $|\ell m\rangle$, then $Y_{\ell m}(\theta,\phi) = Y_{\ell m}(\theta)\, e^{im\phi}$. Find an equation to give $Y_{\ell,-\ell}(\theta)$ up to a multiplicative constant, and a recursion relation to determine the other $Y_{\ell m}$.

4 [5 pts] Complete the calculation begun in class for the Clebsch-Gordon coefficients $(1\,\frac{1}{2}\,\ell\, m|1\,\frac{1}{2}\, m_1\, m_2)$.