Diverges logarithmically at large $k$ as $\int \frac{1}{k^d} \sim \log k$

"UV divergence"

Need to regularize to make any progress

One popular choice: "dimensional regularization" preserves

\[ \int \frac{dk}{k^d} \sim \int \frac{dk}{k^{d-\varepsilon}} \to 0 \]

\[ \varepsilon \to 0 \]

Also: technical trick: how to evaluate integral of form

\[ \frac{1}{AB} = \int \frac{1}{(2\mathbf{A} + (\mathbf{B})^2) (\mathbf{B} - \mathbf{A})^2} \]

\[ = \int \frac{1}{k^2 (k^2 - m^2) + (\mathbf{B} - \mathbf{A})^2 (\mathbf{B} - \mathbf{A})^2} \]

Shifting $\mathbf{k} \to \mathbf{k} + \mathbf{x}$

\[ \mathbf{l}^2 + x(-x)\mathbf{p}^2 - m^2 \]

So...
Last step: Wick rotate $l_0^2 - l^2 = \frac{\Delta}{\Delta + \varepsilon}$

Euclidean continuation

$$f \rightarrow \frac{if}{\sqrt{2}}$$

$$\frac{1}{\sqrt{\Delta + \varepsilon}}$$

So

$$\int \frac{dx}{(2\pi)^d} \exp \left( - \frac{m^2}{2} \right) = \frac{1}{\Gamma(d/2)} \frac{\Gamma(1-d/2)}{\Gamma(1/2)} \frac{\pi^{1-d/2}}{(4\pi)^{d/2}}$$

$d = 4 - \varepsilon$

Change $4 \rightarrow \Theta^2$

Spherical coordinates

$$\int d\Omega \frac{1}{\Delta^{n-1}} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{\Gamma(1/2)} \Delta^{n-1}$$

$$\Gamma(\frac{d}{2}) = \frac{(d-2)\pi^2}{2}$$

$$\Gamma(1-d/2) = \frac{\pi^2}{2}$$

So

$$\int d\Omega = \frac{\pi^2}{2} \frac{\Gamma(1-d/2)}{\Gamma(1/2)} \Delta^{n-1}$$

$d = 4 - \varepsilon$

$$\int d\Omega \frac{1}{\Delta^{n-1}} = \frac{\pi^2}{2} \frac{\Gamma(1-d/2)}{\Gamma(1/2)} \Delta^{n-1}$$

$\Delta \rightarrow \frac{\pi^2}{2} \frac{\Gamma(1-d/2)}{\Gamma(1/2)} \Delta^{n-1}$

$\Theta^2 \rightarrow \text{Mascheroni constant} \approx 0.5772$
How to make sense of this infinity??

First let's consider all such diagrams

\[
\begin{align*}
\mathbb{1} - \Pi &= \Pi(p^2) \\
+ \quad \text{diagram}
\end{align*}
\]

\[
\begin{align*}
\text{Exact propagator} &= \frac{1}{p^2 - m^2} + \frac{1}{p^2 - m^2} \Pi(p^2) \quad \frac{1}{p^2 - m^2} + \frac{1}{p^2 - m^2} \Pi(p^2) \quad \frac{1}{p^2 - m^2} + \frac{1}{p^2 - m^2} \Pi(p^2) \\
\Delta(p^2) &= \frac{1}{p^2 - m^2} \left(1 - \frac{1}{\Pi(p^2)}\right) = \frac{1}{p^2 - m^2 - \Pi(p^2)}
\end{align*}
\]

Corrects propagator!

Note: divergence is just a constant \( \rightarrow \) not mass dependence

Suppose \( m^2 \) in Lagrangian is not true mass but just an infinite bare mass??
Physical mass given by pole in exact propagator

\[ \Delta(p) \mid_{p^2 = m^2} \sim \frac{1}{p^2 - m^2} \]

So could require

\[ m^2 + \Pi(p^2 = m^2) = \frac{2}{m^2} \]

on-shell "Renormalization" Condensation

\[ (\delta m^2 = \delta m_{\text{phys}} + \delta) \]

"counterterm"

So

\[ \delta = -\Pi(p^2 = m^2) = -\Pi(p^2 = m^2) + O(\alpha^4) \]

\[ = -\frac{\delta m_{\text{phys}}}{(4\pi)^2} \int \frac{dx}{x} \left( \frac{1}{x^2} - \ln(x)m^2 + \cdots \right) \]

Infinite counterterm!

More generally, imagine including counterterms for every base parameter in Lagrangian e.g.

\[ \frac{1}{2} \phi \partial \phi \] \[ - \frac{1}{2} Z_m m^2 \phi^2 + gZ_3 \phi^3 \]

\[ Z_{\phi} = 1 + \delta \]

\[ Z_m = 1 + \delta_m \]

\[ Z_3 = 1 + \delta_3 \]

Can cancel off all divergences w/ just 3 counterterms?