1 Physics 613: Problem Set 6 (*due Monday April 29*)

1.1 Attractive and repulsive QCD forces

Recall from class that if we scatter quark of color i against anti-quark of color k, then the t-channel amplitude with color j and ℓ in the final state is proportional to the color factor $T^a_{ji}T^a_{k\ell}$. Let's re-examine this more properly, by considering the representations of $SU(3)_{color}$.

1. As we showed in HW5, the quark antiquark pair can either be in a singlet or an octet representation ($\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}$). Suppose the initial quark-antiquark state is in the singlet representation. Verify explicitly (you can use a computer, e.g. Mathematica or python) that the color factor is $T^a_{ji}T^a_{i\ell} = \frac{4}{3}\delta_{j\ell}$ (summation is implied on a and i here). So color is conserved (singlet to singlet) and the color force is attractive in the color singlet state.

2. Calculate the color factor for the initial state in the octet representation. (Think carefully about how to project onto the octet representation!!) Show that color is again conserved (octet to octet) and show that the color force is repulsive in the color octet state. [Hint: You will need the following identity $T^aT^bT^a = -\frac{1}{2N_c}T^b$.]

1.2 QCD theta term

Prove that the QCD theta term $\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(F_{\mu\nu}F_{\alpha\beta})$ is gauge invariant and a total derivative. [Hint: you will need the following identity for the structure constants $f^{abe}f^{cde} = \frac{2}{N_c}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + d_{ace}d_{bde} - d_{bce}d_{ade}$ where d is a totally *symmetric* tensor (known as the anomaly tensor).]

1.3 Running of α_s and the QCD scale

In class we discussed the one-loop running of the QCD fine structure constant:

$$\alpha_s(Q) = \frac{\alpha_{s0}}{1 + \frac{\alpha_{s0}}{4\pi}\beta \log\frac{Q^2}{m_0^2}} \tag{1}$$

where

$$\beta = 11 - \frac{2}{3}N_f \tag{2}$$

and m_0 is a UV reference scale where $\alpha_s = \alpha_{s0}$. Starting from $\alpha_s = 0.1$ at $m_0 = 1$ TeV, with $N_f = 6$, run the QCD coupling α_s down through each quark threshold to determine

Λ_{QCD} (the scale where α_s appears to blow up at one-loop). You can assume at each quark mass threshold, all that changes is that β takes one smaller value of N_f, and that α_s is continuous through each threshold. You may also take $m_t = 175$ GeV, $m_b = 4$ GeV and $m_c = 1.2$ GeV.

1.4 Numerical PDFs

Download the Mathematica module for numerical PDFs from this link https://ncteq.hepforge.org/mma/PDF_DEMO_v01.zip. Use it to accomplish the following tasks:

1. Make a plot of $xf_i(x)$ vs x for $i = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5$ ($\bar{b}$, $\bar{c}$, $\bar{s}$, $\bar{u}$, $\bar{d}$, d, u, s, c, b) and $i = 21$ (g), at the scale $Q = 100$ GeV. Comment briefly on some of the features of this plot that you find interesting.

2. Verify (just approximately! the PDFs have limited numerical precision) the sum rules

$$\int dx\,(f_u(x) - f_{\bar{u}}(x)) = 2, \qquad \int dx\,(f_d(x) - f_{\bar{d}}(x)) = 1 \tag{3}$$

and

$$\sum_i \int dx\, x f_i(x) = 1 \tag{4}$$

3. Calculate the average momentum fraction carried by the valence quarks (u, d), the sea quarks (all the other quarks), and the gluons, again at $Q = 100$ GeV.

4. In class we studied Drell-Yan ($pp \to \gamma^* \to \ell^+\ell^-$) as a function of the rapidities of the two leptons $y_{3,4}$ and the momentum transfer $\hat{t}$. A simpler version of this is the Drell-Yan differential cross section in terms of the invariant mass-squared of the outgoing leptons M^2, and the rapidity of the virtual photon Y. You can find a formula for this differential cross section in Peskin's book, eq (17.48):

$$\frac{d^2\sigma}{dM^2dY} = \sum_f x_1 f_f(x_1) x_2 f_{\bar{f}}(x_2) \frac{1}{3} Q_f^2 \frac{4\pi\alpha^2}{3M^4} \tag{5}$$

Make a plot of the dilepton invariant mass spectrum at the LHC for M^2 in the range $1-1000$ GeV. What does this plot tell you about the center of mass energies of the colliding partons in the Drell-Yan process?