1 Physics 613: Problem Set 3 (*due May 1*)

1.1 Lie algebra facts

1. Prove that $[T^a, T^b]$ is traceless and anti-Hermitian.

2. Prove that the structure constants f^{abc} are totally antisymmetric.

3. Prove the Jacobi identity $[T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0$

4. Prove that the adjoint commutation relations $[T^a_{adj}, T^b_{adj}] = i f^{abc} T^c_{adj}$ are equivalent to the Jacobi identity.

5. Prove that $[T^a T^a, T^b] = 0$ (therefore $T^a T^a$ must be proportional to the identity in every representation; the proportionality constant is called the Casimir invariant).

1.2 $SU(3)$ generators

Look up the Gell-mann basis of the $SU(3)$ generators and explicitly find all the nonzero structure constants for $SU(3)$ in this basis (please use a computer, e.g. Mathematica, for this, don't do it by hand as it would be way too tedious!)

1.3 $SU(3)$ representations

$SU(3)$ irreducible representations (irreps) are labeled by two integers (n, m) and can be thought of as all multiple-index tensors of the form $A^{i_1 \dots i_n}_{j_1, \dots j_m}$, with indices running from $1, 2, 3$, which are totally *symmetric* in all upper and all lower indices and are *traceless* under contraction of any upper with any lower index. For example, the fundamental rep corresponds to $(1, 0)$ and the anti-fundamental to $(0, 1)$. The adjoint corresponds to $(1, 1)$.

1. Show that $\dim(n, 0) = \dim(0, n) = \frac{1}{2}(n + 2)(n + 1)$.

Irreps can be multiplied (tensor product) and decomposed (direct sum) into smaller irreps by symmetrizing and tracing over indices.

For example we can multiply a fundamental and antifundamental $A^a B_b = C^a_b + \frac{1}{3}\delta^a_b A^c B_c$ where C is the traceless part of AB. In this way we obtain the sum of the adjoint (octet) and the trivial (singlet) representation. We can express this more mathematically as $(1, 0) \otimes (0, 1) = (1, 1) \oplus (0, 0)$, or in terms of the dimensions, $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

One can derive (it takes a bit of thought) the following multiplication rule $(n,0) \otimes (m,0) = (n+m,0) \oplus (n+m-2,1) \oplus (n+m-4,2) \oplus \ldots$ and similarly for $(0,n) \otimes (0,m)$.

2. The (u, d, s) quarks transform in the fundamental of $SU(3)$ and their antiquarks transform in the anti-fundamental of $SU(3)$. Using the facts above, show that the mesons (which are quark anti-quark bound states) must transform in either the adjoint (octet) or the singlet representation.

3. Using the facts above, explain how the baryon octet and decuplet arise from multiplication of $SU(3)$ representations.

1.4 Attractive and repulsive QCD forces

Recall from class that if we scatter quark of color i against anti-quark of color k, then the t-channel amplitude with color j and ℓ in the final state is proportional to the color factor $T^a_{ji} T^a_{k\ell}$. Let's re-examine this more properly, by considering the representations of $SU(3)_{color}$.

1. As we showed in the previous problem, the quark antiquark pair can either be in a singlet or an octet representation ($\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}$). Suppose the initial quark-antiquark state is in the singlet representation. Verify explicitly (you can use a computer, e.g. Mathematica or python) that the color factor is $T^a_{ji} T^a_{i\ell} = \frac{4}{3}\delta_{j\ell}$ (summation is implied on a and i here). So color is conserved (singlet to singlet) and the color force is attractive in the color singlet state.

2. Calculate the color factor for the initial state in the octet representation. (Think carefully about how to project onto the octet representation!!) Show that color is again conserved (octet to octet) and show that the color force is repulsive in the color octet state. [Hint: You will need the following identity $T^a T^b T^a = -\frac{1}{2N_c} T^b$.]

1.5 QCD theta term

Prove that the QCD theta term $\epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta})$ is gauge invariant and a total derivative. [Hint: you will need the following identity for the structure constants $f^{abe} f^{cde} = \frac{2}{N_c}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + d_{ace}d_{bde} - d_{bce}d_{ade}$ where d is a totally *symmetric* tensor (known as the anomaly tensor).]

1.6 Running of α_s and the QCD scale

In class we discussed the one-loop running of the QCD fine structure constant:

$$\alpha_s(Q) = \frac{\alpha_{s0}}{1 + \frac{\alpha_{s0}}{4\pi}\beta \log \frac{Q^2}{m_0^2}} \tag{1}$$

where

$$\beta = 11 - \frac{2}{3}N_f \tag{2}$$

and m_0 is a UV reference scale where $\alpha_s = \alpha_{s0}$. Starting from $\alpha_s = 0.1$ at $m_0 = 1$ TeV, with $N_f = 6$, run the QCD coupling α_s down through each quark threshold to determine Λ_{QCD} (the scale where α_s appears to blow up at one-loop). You can assume at each quark mass threshold, all that changes is that β takes one smaller value of N_f, and that α_s is continuous through each threshold. You may also take $m_t = 175$ GeV, $m_b = 4$ GeV and $m_c = 1.2$ GeV.

1.7 Numerical PDFs

Download the Mathematica module for numerical PDFs from this link https://ncteq.hepforge.org/mma/PDF_DEMO_v01.zip. Use it to accomplish the following tasks:

1. Make a plot of $xf_i(x)$ vs x for $i = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5$ ($\bar{b}$, $\bar{c}$, $\bar{s}$, $\bar{u}$, $\bar{d}$, d, u, s, c, b) and $i = 21$ (g), at the scale $Q = 100$ GeV. Comment briefly on some of the features of this plot that you find interesting.

2. Verify (just approximately! the PDFs have limited numerical precision) the sum rules

$$\int dx\,(f_u(x) - f_{\bar{u}}(x)) = 2, \qquad \int dx\,(f_d(x) - f_{\bar{d}}(x)) = 1 \tag{3}$$

   and

$$\sum_i \int dx\, x f_i(x) = 1 \tag{4}$$

3. Calculate the average momentum fraction carried by the valence quarks (u, d), the sea quarks (all the other quarks), and the gluons, again at $Q = 100$ GeV.

4. In class we studied Drell-Yan ($pp \to \gamma^* \to \ell^+\ell^-$) as a function of the rapidities of the two leptons $y_{3,4}$ and the momentum transfer $\hat{t}$. A simpler version of this is the Drell-Yan differential cross section in terms of the invariant mass-squared of

the outgoing leptons M^2, and the rapidity of the virtual photon Y. You can find a formula for this differential cross section in Peskin's book, eq (17.48):

$$\frac{d^2\sigma}{dM^2 dY} = \sum_f x_1 f_f(x_1) x_2 f_{\bar{f}}(x_2) \frac{1}{3} Q_f^2 \frac{4\pi\alpha^2}{3M^4} \tag{5}$$

Make a plot of the dilepton invariant mass spectrum at the LHC for M^2 in the range $1-1000$ GeV. What does this plot tell you about the center of mass energies of the colliding partons in the Drell-Yan process?