1 Physics 613: Problem Set 2 (due Monday Feb 19)

1.1 Spin and the Dirac Equation

1. Verify that \([L_i, P_j] = i\epsilon_{ijk}P_k\) where \(L = r \times p\) is the angular momentum operator.

2. Verify that \([L_z, H_{Dirac}] = i(\alpha_y P_x - \alpha_x P_y)\). What are \([L_x, H_{Dirac}]\) and \([L_y, H_{Dirac}]\)?

3. Verify that with \(S = \frac{1}{2} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \equiv \frac{1}{2}\Sigma\), the total angular momentum \(J = L + S\) commutes with \(H_{Dirac}\).

1.2 Solutions to the Dirac Equation

In class we introduced the solutions to the Dirac equation \(\psi(x) = u_s(k)e^{-ikx}\) and \(\psi(x) = v_s(k)e^{ikx}\).

1. Verify by plugging into the Dirac equation that \(u_s\) and \(v_s\) satisfy

\[
(k - m)u_s(k) = 0
\]  

(1)

and

\[
(k + m)v_s(k) = 0
\]  

(2)

2. By considering the eigenvalues of \(k - m\) and \(k + m\), prove that there are exactly two independent \(u_s\) and two independent \(v_s\) solutions for every \(k\) (so \(s = 1, 2\)).

3. Show that the helicity operator \(h = \frac{P \cdot \Sigma}{|P|}\) commutes with the Dirac Hamiltonian \(H_{Dirac} = \alpha \cdot P + \beta m\) and find the eigenvalues of \(h\).

4. For momentum in the \(z\) direction (i.e. \(k = (0, 0, k)\)), find explicitly the solutions \(u_s\) and \(v_s\) classified by eigenvalues of the helicity operator.

1.3 Dirac Hamiltonian and charge

1. Substitute the mode expansion

\[
\Psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_s (b_s(k)u_s(k)e^{-ikx} + d_s^*(k)v_s(k)e^{ikx})
\]  

(3)

into the Dirac Hamiltonian

\[
H = \int d^3x \left( -i\bar{\Psi}\gamma^i \partial_i \Psi(x) + m\bar{\Psi}\Psi(x) \right)
\]  

(4)
and, using the canonical commutation relations for \( b_s \) and \( d_s \), derive

\[
H = \sum_s \int \frac{d^3k}{(2\pi)^3 2E_k} E_k (b_s^\dagger(k)b_s(k) + d_s^\dagger(k)d_s(k))
\]  \hspace{1cm} (5)

2. Do the same for the charge operator

\[
Q = \int d^3x \Psi^\dagger \Psi
\]  \hspace{1cm} (6)

and derive

\[
Q = \sum_s \int \frac{d^3k}{(2\pi)^3 2E_k} (b_s^\dagger(k)b_s(k) - d_s^\dagger(k)d_s(k))
\]  \hspace{1cm} (7)