1 Physics 613: Problem Set 1 (due Monday, Feb 5)

1.1 Covariant Formulation of E&M

Maxwell’s equations are

\[ \nabla \cdot \mathbf{E} = 0 \]
\[ \nabla \times \mathbf{B} - \dot{\mathbf{E}} = 0 \]
\[ \nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0 \]
\[ \nabla \cdot \mathbf{B} = 0 \]  

(1)

with \( \mathbf{E} \) and \( \mathbf{B} \) given in terms of the scalar and vector potential as:

\[ \mathbf{E} = -\nabla \Phi - \dot{\mathbf{A}} \]
\[ \mathbf{B} = \nabla \times \mathbf{A} \]  

(2)

We can combine \( \Phi \) and \( \mathbf{A} \) into the Lorentz-covariant gauge potential \( A_\mu = (\Phi, \mathbf{A}) \) and define the gauge invariant field strength \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

1. Show how \( F_{\mu\nu} \) contains the \( \mathbf{E} \) and \( \mathbf{B} \) fields.

2. Show how the covariant form of Maxwell’s equations \( \partial_\mu F^{\mu\nu} = 0 \), together with the relations between the gauge potential and the \( \mathbf{E} \) and \( \mathbf{B} \) fields (2), imply the usual component form of Maxwell’s equations (1).

3. In class we argued that \( F_{\mu\nu}F^{\mu\nu} \) was the unique object which satisfies the following requirements: Lorentz invariant, gauge invariant, quadratic in \( A_\mu \) and second order in derivatives. There is one other possible object that satisfies these requirements: \( \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \), where \( \epsilon^{\mu\nu\rho\sigma} \) is the totally antisymmetric 4-index tensor. Show that this term (called the \( \theta \)-term) is in fact a total derivative.

1.2 Canonical Quantization of a Scalar Field

A much simpler field theory than Maxwell theory is that of a real scalar field \( \phi(x) \), with Lagrangian:

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \]  

(3)

1. Derive the equation of motion for \( \phi \):

\[ \Box \phi = m^2 \phi \]  

(4)

This is called the Klein-Gordon equation.
2. Derive the conjugate momentum $\Pi = \dot{\phi}$ and show that the Hamiltonian density is
\[ H = \frac{1}{2} \Pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2\phi^2 \] (5)

3. The mode expansion for $\phi$ is
\[ \phi(x) = \int \frac{d^3k}{(2\pi)^32E_k} \left( a_k e^{ikx} + a_k^\dagger e^{-ikx} \right) \] (6)
Remember that $kx = k^\mu x_\mu = E_k t - k \cdot x$. How are $E_k$ and $k$ related?

4. Show that if $a_k$ and $a_k^\dagger$ satisfy the commutation relations of an infinite set of simple harmonic oscillators
\[ [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 \]
\[ [a_k, a_{k'}^\dagger] = (2\pi)^3(2E_k)\delta^3(k - k') \] (7)
then one obtains the canonical commutation relations for $\phi$ and $\Pi$:
\[ [\phi(x, t), \phi(x', t)] = [\Pi(x, t), \Pi(x', t)] = 0 \]
\[ [\phi(x, t), \Pi(x', t)] = i\delta^3(x - x') \] (8)

5. Plug the mode expansion (6) into the Hamiltonian density (5) and show that the Hamiltonian reduces to that of an infinite set of decoupled simple harmonic oscillators (with energy $E_k$):
\[ H = \int d^3x \mathcal{H} = \int \frac{d^3k}{(2\pi)^32E_k} E_k a_k^\dagger a_k \] (9)
up to an overall (infinite) zero point energy.

### 1.3 Dirac Matrices

In class we derived the Dirac equation by starting from the requirement that there exist $N \times N$ Hermitian matrices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta$ satisfying
\[ (p^2 + m^2)\mathbb{1}_N = (\alpha \cdot p + \beta m)^2 \] (10)
Here we will verify a number of the steps that we skipped over.

1. Verify that (10) implies $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1$ and $\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta\} = 0$ (for $i \neq j$).
2. Use part 1 to prove that the $\alpha$ and $\beta$ matrices must be traceless and have eigenvalues $\pm 1$. (Together these imply that $N$ must be even.)

3. Prove that there is no solution to (10) with $N = 2$.

4. For $N = 4$ verify that $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$ satisfy (10).

5. Verify that $\gamma^0 = \beta$ and $\gamma^i = \beta \alpha^i$ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.