HW #5 solutions (2026)

(1.) (a) Uniform $m \Rightarrow \left(\frac{dm}{dx}\right)^2 = 0$, we just need to minimize $V(m) = \frac{a}{2}m^2 + \frac{b}{4}m^4$.

For $\underset{a<0}{T<T_c}$, $\underbrace{\frac{dV}{dm}}_{= am + bm^3} = 0$ yields

$$m = 0, \quad m^2 = -\frac{a}{b} = \frac{|a|}{b}.$$

The minima are given by $m = \pm m_0$, where $m_0 = \sqrt{\frac{|a|}{b}}$.

Indeed, $V(m_0) = V(-m_0) = \frac{a}{2}\left(-\frac{a}{b}\right) + \frac{b}{4}\frac{a^2}{b^2} =$

$$= -\frac{a^2}{4b} < V(0) = 0.$$

(b) Use Euler-Lagrange equation:

$$\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial (m')} = \frac{\partial \mathcal{L}}{\partial m},$$

$m' = \frac{dm}{dx}$

where the "Lagrangian" is the free energy density:

$$F[m] = \int_{-\infty}^{\infty} dx\, \mathcal{L}(m, m'; x).$$

We obtain: $$c \underbrace{\frac{d^2 m}{dx^2}}_{\text{"m" for brevity}} = am + bm^3. \qquad (*)$$

(c) Multiply Eq. (*) by m':

$$c\, m'' m' = (a m + b m^3)\, m', \text{ or}$$

$$c \frac{d}{dx}\left[\frac{(m')^2}{2}\right] = \frac{d}{dx}\left[\frac{a}{2} m^2 + \frac{b}{4} m^4\right].$$

Integrate: $\frac{c}{2}(m')^2 = \frac{a}{2} m^2 + \frac{b}{4} m^4 + \underbrace{C_1}_{\text{const}}$

At $x \to \pm\infty$, $m \to \pm \underset{\substack{\uparrow \\ \text{from (a)}}}{m_0}$ and $m' \to 0$.

Then $C_1 = -\underbrace{\left(\frac{a}{2} m_0^2 + \frac{b}{4} m_0^4\right)}_{V(m_0) \,=\, -\frac{a^2}{4b} \text{ from (a)}}.$

Thus, $\frac{c}{2}(m')^2 = \frac{a}{2} m^2 + \frac{b}{4} m^4 + \frac{a^2}{4b} =$

$$= \frac{b}{4} \underbrace{(m^2 - m_0^2)^2}_{=(m_0^2 - m^2)^2}.$$

Indeed, $\frac{b}{4}\left(m^4 - 2 \underbrace{m_0^2}_{-\frac{a}{b}} m^2 + \underbrace{m_0^4}_{\frac{a^2}{b^2}}\right) =$

$$= \frac{b}{4} m^4 + \frac{a}{2} m^2 + \frac{a^2}{4b}, \text{ as desired.}$$

We need to solve: $\frac{dm}{dx} = \sqrt{\frac{b}{2c}}\,(m_0^2 - m^2)$, or

$$\int \frac{dm}{m_0^2 - m^2} = \int \sqrt{\frac{b}{2c}}\, dx$$

Finally, $\frac{1}{m_0} \tanh^{-1}\left(\frac{m}{m_0}\right) = \sqrt{\frac{b}{2c}}\, x$, or

$$m(x) = m_0 \tanh\left(\underbrace{\sqrt{\frac{b}{2c}}\, m_0}\, x\right).$$

$$\sqrt{\frac{b}{2c}} \underbrace{\sqrt{\frac{|a|}{b}}}_{m_0 \text{ from (a)}} = \frac{1}{\sqrt{2}} \sqrt{\frac{|a|}{c}} \Rightarrow \underline{\underline{\xi = \sqrt{\frac{c}{|a|}}}}.$$

So, $\begin{cases} m_0 = \sqrt{\frac{|a|}{b}}, \\ \xi = \sqrt{\frac{c}{|a|}} \end{cases}$ and $m(x) = m_0 \tanh\left(\frac{x}{\sqrt{2}\,\xi}\right)$, as desired.

Since $a = \tilde{a}(T - T_c)$, $\xi \sim \frac{1}{\sqrt{T_c - T}}$.

ξ is the correlation length;

$\xi \to \infty$ as $T \to T_c^-$, signifying that the domain walls become very broad.

(2.) (a) Represent each philosopher by

$\sigma_i = \{-1, +1\}$; $i = 1, \dots, N$

↑ spin   ↑ # spins

Circular table → periodic BCs

Note that if $\sigma_i = +1$, $\sigma_{i-1} = \sigma_{i+1} = -1$ are not allowed.

(b) <u>$N=2$:</u>

$$\left.\begin{array}{cc} - & - \\ - & + \\ + & - \end{array}\right\} M_N = 3$$

<u>$N=3$:</u>

$$\left.\begin{array}{ccc} - & - & - \\ + & - & - \\ - & + & - \\ - & - & + \end{array}\right\} M_N = 4$$

<u>$N=4$:</u>

$$\left.\begin{array}{l} - - - - \\ \text{4 states with one } + \\ - + - + \\ + - + - \end{array}\right\} M_N = 7$$

<u>$N=5$:</u>

$$\left.\begin{array}{l} - - - - - \\ \text{5 states with one } + \\ \left.\begin{array}{l} + - + - - \\ + - - + - \\ - + - + - \\ - + - - + \\ - - + - + \end{array}\right\} \text{two } +\text{'s} \end{array}\right\} M_N = 11$$

(c) Note that

$$\underbrace{M_4}_{=7} = M_3 + M_2 = 4+3,$$

$$\underbrace{M_5}_{=11} = M_4 + M_3 = 7+4.$$

In general, $M_N = M_N(-1) + M_N(+1)$.

Now,

$$M_{N+1}(-1) = \underbrace{M_N}_{\substack{\text{just add} \\ -1 \text{ @ pos. } N+1 \\ \text{to any } N\text{-spin} \\ \text{configuration}}} + \underbrace{M_{N-1}(+1)}_{\substack{\underbrace{\cdots\uparrow}_{N-1} \\ \text{becomes} \\ \underbrace{\cdots\uparrow\downarrow\uparrow}_{N+1}}}$$

$$M_{N+1}(+1) = \underbrace{M_{N-1}(-1)}_{\substack{\underbrace{\cdots\downarrow}_{N-1} \\ \text{becomes} \\ \underbrace{\cdots\downarrow\uparrow\downarrow}_{N+1}}}$$

Therefore, $M_{N+1} = M_N + M_{N-1}$. $\quad (*)$

(d) The contribution of 2 neighboring spins to $\mathcal{H}$ is

$$\Delta E(\sigma, \sigma') = -J\sigma\sigma' - \frac{h}{2}(\sigma + \sigma').$$

with $J = -1$, $h = -2$ we obtain:

σ / σ'	$+1$	-1
$+1$	3	-1
-1	-1	-1

$\Leftarrow$ table of $\Delta E(\sigma, \sigma')$ values

All states that are allowed in the philosopher problem have the same ground-state energy $E_0(N) = -\underbrace{N}_{\substack{\#\text{ nnb pairs in} \\ \text{1D chain}}}$.

The forbidden states have higher energies. Thus, it suffices to count the # ground states to find M_N.

(e) In the $\beta \to$ limit,

$$Z_N \to \sum_{\substack{\text{ground} \\ \text{states}}} e^{-\beta E_0(N)} = e^{-\beta E_0(N)} M_N.$$

↑ excited states do not contribute

Thus, $\quad M_N = \lim_{\beta \to \infty} e^{\beta E_0(N)} Z_N.$

Now, recall that

$$Z_N = \lambda_0^N + \lambda_1^N \quad , \quad \text{where}$$

$$\lambda_{0,1} = e^{\beta J} \cosh(\beta H) \pm \sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}} .$$

With $J=-1$, $h=-2$ we have:

$$\lambda_{0,1} = e^{-\beta} \cosh(2\beta) \pm \sqrt{e^{-2\beta} \sinh^2(2\beta) + e^{2\beta}} .$$

In the $\beta \to \infty$ limit,

$$\lambda_{0,1} \to e^{-\beta} \frac{e^{2\beta}}{2} \pm \sqrt{e^{-2\beta} \frac{e^{4\beta}}{4} + e^{2\beta}} =$$

$$= e^{\beta} \frac{1}{2} \pm e^{\beta} \frac{\sqrt{5}}{2} = e^{\beta} \left(\frac{1 \pm \sqrt{5}}{2} \right) .$$

Finally,

$$M_N \xrightarrow[\beta \to \infty]{} e^{-\beta N} \left[e^{\beta N} \left(\frac{1+\sqrt{5}}{2} \right)^N + e^{\beta N} \left(\frac{1-\sqrt{5}}{2} \right)^N \right] =$$

$$= \left(\frac{1+\sqrt{5}}{2} \right)^N + \left(\frac{1-\sqrt{5}}{2} \right)^N .$$

Note that $\overbrace{M_0 = 2, \quad M_1 = 1}^{\text{"non-physical states"}}$,

$$M_2 = \frac{1 + 2\sqrt{5} + 5}{4} + \frac{1 - 2\sqrt{5} + 5}{4} = 3 \text{, etc.}$$

Finally, note that

$$M_N + M_{N-1} = \left(\frac{1+\sqrt{5}}{2}\right)^N + \left(\frac{1-\sqrt{5}}{2}\right)^N +$$

$$+ \left(\frac{1+\sqrt{5}}{2}\right)^{N-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{N-1} =$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{N-1} \underbrace{\left(\frac{3+\sqrt{5}}{2}\right)}_{\left(\frac{1+\sqrt{5}}{2}\right)^2} + \left(\frac{1-\sqrt{5}}{2}\right)^{N-1} \underbrace{\left(\frac{3-\sqrt{5}}{2}\right)}_{\left(\frac{1-\sqrt{5}}{2}\right)^2} =$$

$= M_{N+1}$, consistent with (*).