HW #4 solutions (2026)

(1.) Note that $p_1 = p_2$ at thermal equilibrium.

(a) $T = 0$

Degenerate Fermi gas: $\mathcal{U} = \frac{3}{5} N \epsilon_F$,

where $\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{(2s+1)} n \right)^{2/3}$.

$n = \frac{N}{V} = \frac{1}{v}$ particle density

const (V, s) ← spin

Then $p = -\left(\frac{\partial \mathcal{U}}{\partial V}\right)_S = A \frac{n^{5/3}}{(2s+1)^{2/3}}$

S ← entropy

Finally, $p_1 = p_2$ yields

$$\left(\frac{n_1}{n_2}\right)^{5/3} = \frac{(2s_1+1)^{2/3}}{(2s_2+1)^{2/3}}, \text{ or}$$

$$\frac{V_2}{V_1} = \frac{(2s_1+1)^{2/5}}{(2s_2+1)^{2/5}}.$$

(b) $T = \infty$

Classical ideal gas: $\frac{n_1}{n_2} = \frac{V_2}{V_1} = 1$.

$p = n k_B T$

② 2D ideal Bose gas

(a) $\log \Sigma = - \sum_{\vec{p}} \log(1 - \underset{\substack{\| \\ e^{\beta\mu}}}{z}\, e^{-\beta\epsilon_{\vec{p}}})$ $\ominus$

$e^{\beta\mu}$, fugacity

$$\ominus \; -\frac{2\pi L^2}{h^2}\int_0^\infty dp\, p\, \underbrace{\log(1 - z e^{-\beta\epsilon_{\vec{p}}})}_{\text{expand in powers of } z}, \text{ where}$$

$$\epsilon_{\vec{p}} = \frac{p^2}{2m} \; ; \quad L^2 = \underset{\text{area}}{A}$$

Further more,

$$\log \Sigma = + \frac{2\pi L^2}{h^2} \sum_{k=1}^{\infty} \frac{1}{k}\int_0^\infty dp\, p \left(z e^{-\frac{\beta p^2}{2m}}\right)^k \ominus$$

$$-\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ominus \; \frac{2\pi L^2}{h^2}\sum_{k=1}^{\infty}\frac{z^k}{k}\underbrace{\int_0^\infty dp\, p\, e^{-\frac{k\beta p^2}{2m}}}_{\frac{m}{k\beta}} = \frac{2\pi L^2}{h^2}\frac{m}{\beta}\underbrace{\sum_{k=1}^{\infty}\frac{z^k}{k^2}}_{g_2(z)} \diamond$$

$$\diamond \; \frac{L^2}{2\pi\hbar^2} m k_B T\, g_2(z).$$

(b) Recall that

$$\frac{p}{k_B T} = \frac{\log \Sigma}{L^2}, \quad \text{then}$$

$$\frac{p}{k_B T} = \underbrace{\frac{m k_B T}{2\pi \hbar^2}}_{\lambda^{-2}} g_2(z) = \underline{\underline{\frac{g_2(z)}{\lambda^2}}}.$$

(c) $$\underset{\substack{\uparrow \\ \text{part. density}}}{n} = \frac{N}{L^2} = z \frac{\partial}{\partial z}\left(\frac{\log \Sigma}{L^2}\right) = z \frac{\partial}{\partial z}\left(\frac{g_2(z)}{\lambda^2}\right) =$$

$$= \lambda^{-2} z \sum_{k=1}^{\infty} \frac{z^{k-1}}{k} = \lambda^{-2} \underbrace{\sum_{k=1}^{\infty} \frac{z^k}{k}}_{-\log(1-z)} = \underline{\underline{-\frac{\log(1-z)}{\lambda^2}}}.$$

Finally, $$\log(1-z) = -n\lambda^2,$$

$$z = 1 - e^{-\lambda^2 n}, \quad \text{or}$$

$$\underset{\substack{\uparrow \\ \mu(n,T)}}{\mu} = \underline{\underline{k_B T \log(1 - e^{-\lambda^2 n})}}.$$

3. The allowed values of

$$\vec{k} = \frac{2\pi}{L}(n_1, \ldots, n_d), \quad \text{where}$$

$$n_i = 0, \pm 1, \pm 2, \ldots \qquad i = 1, \ldots, d$$

Then $\epsilon(\vec{k}) = c|\vec{k}| = \dfrac{2\pi c}{L}\sqrt{\sum_{i=1}^{d} n_i^2}$.

(a) As before,

$$\log \mathcal{Z} = -\sum_{\vec{k}} \log\left(1 - e^{-\beta(\epsilon(\vec{k}) - \mu)}\right) =$$

$$= -\frac{\overset{\sim L^d}{V}}{(2\pi)^d} \underbrace{\int d^d\vec{k}}_{\int_0^\infty dk\, k^{d-1} \underbrace{S_d}_{\text{area of the unit sphere in } d \text{ dimensions}}} \log\left(1 - e^{-\beta(\epsilon(k) - \mu)}\right).$$

(b) $$\langle N \rangle = \left(\frac{\partial \log \mathcal{Z}}{\partial(\beta\mu)}\right)_{V,\beta} = \sum_{\vec{k}} \frac{1}{e^{\beta(\epsilon(k) - \mu)} - 1}, \quad \text{where}$$

$$\sum_{\vec{k}} \Rightarrow \frac{V}{(2\pi)^d}\int d^d\vec{k} \Rightarrow \frac{V}{(2\pi)^d} S_d \int_0^\infty dk\, k^{d-1} \quad \text{as in (a).}$$

So, $$\langle N \rangle = \left(\frac{L}{2\pi}\right)^d S_d \int_0^\infty dk\, k^{d-1} \frac{1}{e^{\beta(ck - \mu)} - 1}.$$

(c) Consider

$$\langle N \rangle = \left(\frac{L}{2\pi}\right)^d S_d \left(\frac{k_B T}{c}\right)^d \underbrace{\int_0^\infty dx \, \frac{x^{d-1}}{z^{-1} e^{x} - 1}}_{\equiv I(z)} ,$$

where $z = e^{\beta\mu}$.

Thus, $\underbrace{\frac{\langle N \rangle}{V}}_{\text{average density of particles}} \sim I(z)$.

Note that $I(z) \leq I(1) = \int_0^\infty dx \, \frac{x^{d-1}}{e^{x} - 1}$.

$I(1)$ diverges at $d=1$ and converges for $d \geq 2$. Since $I(1)$ represents the upper bound for the density of excited particles, BE condensation can happen for $d = 2, 3, 4, \dots$ ('extra' particles will go into the condensate), but not for $d=1$ where $I(1) \to \infty$.