Final (2026)

1. <u>Virial expansion</u>

Consider dilute classical gas with pairwise interactions $u(r_{ij}) = u(|\vec{r}_i - \vec{r}_j|)$ between particles.

(a) Use the virial expansion machinery to show that $p \simeq nk_BT[1 + B_2 n]$ ← gas pressure with the lowest-order correction. Find the expression for B_2.

<u>Hint</u>: Define $f(r_{ij}) = e^{-\beta u(r_{ij})} - 1$ to facilitate the expansion.

(b) Compute B_2 for the hard-sphere potential:

$$u(r) = \begin{cases} \infty, & r < 2a \\ 0, & r > 2a \end{cases}$$

Comment on its sign and its effect on the pressure compared to the ideal gas.

(c) Discuss the effect of attractive pairwise interactions on the gas pressure (no need to evaluate B_2 explicitly here).

Blackbody radiation

(2.) Consider a photon gas in a blackbody cavity at temperature T.

(a) Compute the grand potential

$\Omega = -k_B T \log Q$ for this system

Q: grand-canonical partition function

<u>Hint</u>: Use $\int_0^\infty dx\, x^2 \log(1-e^{-x}) = -\frac{\pi^4}{45}$.

(b) Find the pressure p and discuss its T-dependence.

(c) Find the relation between p and $u = \frac{U}{V}$, the energy density.

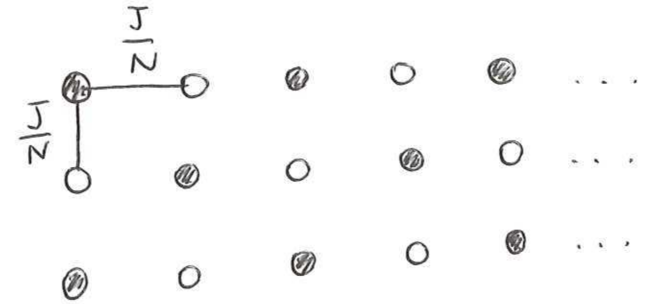
(d) Find the total entropy of the photon gas in terms of U and T.

③ Mean-field Ising model

Consider a bi-partite lattice with two sublattices described by the Hamiltonian:

$$H = -\frac{2J}{N} \sum_{\substack{i\in A,\\ j\in B}} S_i S_j - h_A \sum_{i\in A} S_i - h_B \sum_{j\in B} S_j ,$$

where $S_i = \pm 1$ and each sublattice has $\frac{N}{2}$ spins:



$N_A = N_B = \frac{N}{2}$

Each A spin interacts with all B spins, and vice versa.

(a) Write down mean-field equations for

$$\begin{cases} m_A = \frac{2}{N} \sum_{i\in A} S_i , \\ m_B = \frac{2}{N} \sum_{j\in B} S_j . \end{cases}$$

sublattice magnetizations

(b) In the $h_A = 0$, $h_B = 0$ case, find T_c, the critical temperature of the ferromagnetic transition.

Hint: use mean-field equations from part (a).

(c) In the staggered-field case,

$$\begin{cases} h_A = h, \\ h_B = -h \end{cases}$$

find staggered magnetization, $m_s = \frac{m_A - m_B}{2}$, around T_c, as a function of h.

(d) Find staggered isothermal susceptibility,

$$\chi_s = \left(\frac{\partial m_s}{\partial h}\right)_T,$$

in the vicinity of the critical point ($T = T_c$).

Discuss the behavior of χ_s at T_c.