HW #3 solutions

1. 3.18

N spins in a magnetic field H:

$$E(n_1, n_2, \dots, n_N) = -\underset{\substack{\uparrow \\ \text{magnetic moment}}}{\mu} H \sum_{i=1}^{N} n_i \quad , \quad n_i = \pm 1 \, , \, \forall i$$

(a) β, H, N ensemble:

use $$Q = \sum_{n_1, \dots, n_N} e^{-\beta E(n_1, \dots, n_N)} = \prod_{i=1}^{N} \sum_{n=\pm 1} e^{\beta \mu H n} =$$

$$= (\underbrace{2\cosh(\beta\mu H)}_{e^{\beta\mu H} + e^{-\beta\mu H}})^N .$$

Then $$\underbrace{\langle E \rangle}_{\text{internal energy}} = \frac{\partial \log Q}{\partial(-\beta)} = N\mu H \frac{e^{-\beta\mu H} - e^{\beta\mu H}}{e^{-\beta\mu H} + e^{\beta\mu H}} =$$

$$= -N\mu H \tanh(\beta\mu H).$$

(b) $$S = \frac{\langle E \rangle - \overset{= -\beta^{-1}\log Q}{A}}{T} = k_B \log Q + k_B \beta \langle E \rangle =$$

$$= N k_B \log(e^{\beta\mu H} + e^{-\beta\mu H}) - N\mu H k_B \beta \tanh(\beta\mu H).$$

(c) as $T \to 0$ $(\beta \to \infty)$, we get

$\tanh(\beta\mu H) \to 1$.

Then $\langle E \rangle \underset{T\to 0}{\to} \underbrace{-N\mu H}$

all spins at +1, aligned with the field

$$S \underset{T\to 0}{\to} Nk_B(\beta\mu H) - \beta\mu H(Nk_B) = 0.$$

The entropy is 0 at $T=0$.

2. 3.19

(a) $\langle M \rangle = \langle \sum_{i=1}^{N} \mu n_i \rangle = -\frac{1}{H} \langle \sum_{i=1}^{N} (-\mu H n_i) \rangle =$

$$= -\frac{\langle E \rangle}{H} = \underline{\underline{N\mu \tanh(\beta\mu H)}}.$$

(b) $\delta M = M - \langle M \rangle$

Consider $\langle (\delta M)^2 \rangle = \langle M^2 \rangle - \langle M \rangle^2 =$

$$= \frac{1}{Q} \underbrace{\frac{\partial^2 Q}{\partial(\beta H)^2}} - \frac{1}{Q^2}\left(\frac{\partial Q}{\partial(\beta H)}\right)^2 \Leftrightarrow$$

$$\frac{\partial}{\partial(\beta H)}\left[N\mu Q \tanh(\beta\mu H)\right] =$$

$$= N\mu\left[\frac{\partial Q}{\partial(\beta H)} \tanh(\beta\mu H) + Q \frac{\partial}{\partial(\beta H)} \tanh(\beta\mu H)\right]$$

$$\Leftrightarrow N\mu \tanh(\beta\mu H) \underbrace{\frac{\partial \log Q}{\partial(\beta H)}}_{N\mu \tanh(\beta\mu H)} + N\mu^2 \overbrace{\underbrace{\tanh'(\beta\mu H)}_{1-\tanh^2(\beta\mu H)}}^{-N^2\mu^2\tanh^2(\beta\mu H)} =$$

$$= N\mu^2\left[(N-1)\tanh^2(\beta\mu H) + 1\right] - N^2\mu^2 \tanh^2(\beta\mu H) =$$

$$= \underline{\underline{N\mu^2\left[1-\tanh^2(\beta\mu H)\right]}}.$$

Furthermore,

$$\left.\frac{\partial \langle M\rangle}{\partial H}\right|_{\beta,N} = N\mu \frac{\partial}{\partial H}\tanh(\beta\mu H) =$$

$$= N\mu^2\beta \frac{\partial}{\partial(\beta\mu H)}\tanh(\beta\mu H) =$$

$$= \beta N\mu^2\left[1-\tanh^2(\beta\mu H)\right].$$

Thus, $$\left.\frac{\partial\langle M\rangle}{\partial H}\right|_{\beta,N} = \underline{\underline{\beta\langle(\delta M)^2\rangle}}.$$

(c) as $T\to 0$ $(\beta\to\infty)$: $\tanh(\beta\mu H)\to 1$.

Hence $$\langle M\rangle \xrightarrow[T\to 0]{} N\mu,$$

$$\langle(\delta M)^2\rangle \xrightarrow[T\to 0]{} N\mu^2[1-1] = 0.$$

at $T=0$, all spins are aligned and there are no fluctuations.

(3.) 3.20

If $M = \text{const}$, $E = -HM$ and the ensemble is microcanonical

$\underbrace{HM}_{\text{const as well if } H=\text{const}}$

Then $M = \mu \sum_{i=1}^{N} n_i = \mu(\underbrace{n_+}_{\#\text{spins up}} - \underbrace{n_-}_{\#\text{spins down}})$ $=$

$n_+ + n_- = N$

$= \mu(2n_+ - N)$.

$E = -H\mu(2n_+ - N)$.

} functions of n_+

The # states with energy E is:

$$\Omega(E) = \frac{N!}{n_+!(N-n_+)!}.$$

Recall that $\frac{S}{k_B} = \log \Omega(E)$ and

$$\beta = \frac{1}{k_B} \frac{\partial S}{\partial E}\bigg|_{N,H} = \frac{\partial \log \Omega}{\partial E}\bigg|_{N,H} = \frac{\partial \log \Omega}{\partial n_+} \underbrace{\frac{\partial n_+}{\partial E}}_{-\frac{1}{2\mu H}}\bigg|_{N,H} =$$

$$= \frac{1}{2\mu H} \underbrace{\frac{\partial}{\partial n_+}\left(\log(n_+!(N-n_+)!)\right)}_{}.$$

$N \gg 1$

$$\frac{\partial}{\partial n_+}\left[n_+ \log n_+ - n_+ + (N-n_+)\log(N-n_+) - (N-n_+)\right]$$

(»)

$$\Leftrightarrow \log n_+ + 1 - \log(N-n_+) - 1 = \log \frac{n_+}{N-n_+}.$$

Thus, $\underline{2\beta\mu H = \log \frac{n_+}{N-n_+}}$, or

$$e^{-2\beta\mu H} = \frac{N}{n_+} - 1,$$

$$\underline{\underline{n_+ = \frac{N}{1+e^{-2\beta\mu H}}.}}$$

Moreover,

$$M = \mu\left(\frac{2}{1+e^{-2\beta\mu H}} - 1\right)N =$$

$$\underline{\underline{= \mu N \tanh(\beta\mu H).}}$$ ← as before

$$E = -HM = -\mu N H \tanh(\beta\mu H).$$

The rest of thermodynamics should follow

Finally, we can use $n_+ = \frac{1}{2}\left[\frac{M}{\mu} + N\right]$ to obtain

$$\beta H = \frac{1}{2\mu}\log\left(\frac{\frac{1}{2}\left[\frac{M}{\mu}+N\right]}{N - \frac{1}{2}\left[\frac{M}{\mu}+N\right]}\right) = \frac{1}{2\mu}\log\left(\frac{M+N\mu}{N\mu - M}\right),$$

where $\underline{\underline{M = -\frac{E}{H}}}$

implicit eq'n for $H = H(E,N)$