Homework 3 - Solutions PHYS502 2018

Problem 1.

Use the WKB approximation to determine the bound state energies of

$$V(x) = \begin{cases} \dfrac{V_0}{a}|x|, & |x| \le a \\ V_0 = \dfrac{1}{m}\left(\dfrac{h}{a}\right)^2, & |x| \ge a. \end{cases}$$

The turning points are $x_1 = Ea/V_0$, $x_2 = -x_1$. We write

$$\phi = \frac{1}{\hbar}\int_{x_1}^{x_2} \sqrt{2m(E-V)}\,\mathrm{d}x = \left(n+\frac{1}{2}\right)\pi \ .$$

For the given problem

$$\frac{2}{\hbar}\int_0^{Ea/V_0} \sqrt{2m\left(E - \frac{V_0}{a}x\right)}\,\mathrm{d}x = \left(n+\frac{1}{2}\right)\pi \ .$$

Evaluating the integral we obtain

$$\frac{2a}{3\hbar m V_0}(2mE_n)^{3/2} = \left(n+\frac{1}{2}\right)\pi \ .$$

That is,

$$E_n = \left(\frac{3V_0 h}{8a\sqrt{2m}}\right)^{2/3}\left(n+\frac{1}{2}\right)^{2/3} \ .$$

To get the maximum bound state $E_{n,\max} \le V_0$ which gives

$$V_0 = \left(\frac{3V_0 h}{8a\sqrt{2m}}\right)^{2/3}\left(n_{\max}+\frac{1}{2}\right)^{2/3} \ .$$

Substituting $V_0 = h^2/(ma)$ we get

$$V_0 = \left(\frac{3V_0^{3/2}}{8\sqrt{2}}\right)^{2/3}\left(n_{\max}+\frac{1}{2}\right)^{2/3} \ .$$

Or $n_{\max} = \dfrac{8\sqrt{2}}{3} - \dfrac{1}{2} = 3.27$. That is, $n_{\max} = 3$. Thus, there are four bound states with $n = 0, 1, 2, 3$. They are

$$\begin{aligned} E_0 &= \left(\frac{3}{16\sqrt{2}}\right)^{2/3} V_0 \ , \quad E_1 = 3^{2/3}E_0 \ , \\ E_2 &= 5^{2/3}E_0 \ , \quad E_3 = 7^{2/3}E_0 \ . \end{aligned}$$

Problem 2.

A neutron falling in a gravitational field and bouncing-off a horizontal mirror exhibits quantized energy levels. The potential of the problem is $V(z) = mgz$ for $z > 0$ and ∞ for $z < 0$. Applying the WKB method obtain the energy levels of the system.

For the present problem we cannot use the expression

$$\frac{1}{\hbar}\int_{x_1}^{x_2} \sqrt{2m(E - V(x)}\,\mathrm{d}x = \left(n + \frac{1}{2}\right)\pi.$$

Because the above formula was derived under the assumption that the WKB wave function leaks into the $x < x_1$ region. In the given problem the wave function must strictly vanish at $x \leq x_1 = 0$. One can use the potential $V(x) = mg|x|$, $-\infty < x < \infty$ and consider only the odd-parity solutions. The turning points are $x_1 = -E/(mg)$, $x_2 = E/(mg)$. Hence,

$$\int_{x_1}^{x_2} \sqrt{2m(E - mg|x|}\,\mathrm{d}x = \left(n_{\mathrm{odd}} + \frac{1}{2}\right)\pi\hbar, \quad n_{\mathrm{odd}} = 1, 3, \cdots .$$

We write

$$2\int_{0}^{E/mg} \sqrt{2m(E - mgx}\,\mathrm{d}x = \left(n_{\mathrm{odd}} + \frac{1}{2}\right)\pi\hbar, \quad n_{\mathrm{odd}} = 1, 3, \cdots .$$

We obtain the value of the integral as $\dfrac{3}{m^2 g}(2m)^{3/2}E^{3/2}$. Then

$$E_n = \left[3\left(n - \frac{1}{4}\right)\pi\right]^{2/3} (mg^2\hbar^2/8)^{1/3} .$$

Problem 3.

Estimate the ground state of the infinite-well (one-dimensional box) problem defined by

$$V = \begin{cases} 0, & \text{for } |x| < L \\ \infty, & \text{for } |x| > L, \end{cases}$$

using the trial eigenfunction $\phi = |L|^\alpha - |x|^\alpha$ with α the trial parameter and compare it with the exact energy value.

We obtain

$$\begin{aligned} \langle E \rangle &= \frac{-\dfrac{\hbar^2}{2m}\displaystyle\int_{-L}^{L}\left(\phi^* \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)\mathrm{d}x}{\displaystyle\int_{-L}^{L}\phi^*\phi\,\mathrm{d}x} \\ &= \frac{\dfrac{\hbar^2}{2m}\alpha(\alpha-1)\displaystyle\int_0^L \left(L^\alpha - x^\alpha\right)x^{\alpha-2}\,\mathrm{d}x}{\displaystyle\int_0^L \left(L^{2\alpha} - 2L^\alpha x^\alpha + x^{2\alpha}\right)\mathrm{d}x} \\ &= \frac{(\alpha+1)(2\alpha+1)}{2\alpha-1}\left(\frac{\hbar^2}{4mL^2}\right) . \end{aligned}$$

From $\partial\langle E\rangle/\partial\alpha = 0$ we get $\alpha = (1 \pm \sqrt{6}\,)/2$. Since α has to be positive for physically acceptable solution we choose $\alpha = (1 + \sqrt{6}\,)/2 \approx 1.72$. Then using $E_{\text{exact}} = \hbar^2\pi^2/(8mL^2)$ we obtain

$$\langle E \rangle = \frac{2.72 \times 4.44 \times 2 \times E_{\text{exact}}}{2.44 \times \pi^2} = 1.003 E_{\text{exact}} .$$

The percentage of error is 0.3%.

Problem 4.

Consider the triangular potential $V(x) = \begin{cases} Fx, & \text{if } x > 0 \\ \infty, & \text{if } x < 0. \end{cases}$ This is used as a model for an electron trapped on the surface of liquid helium by an electric field due to two capacitor plates bracketing the helium and vacuum above it or for the MOSFET. Applying the variational method calculate ground state energy. Use the trial eigenfunction as $\phi = x\mathrm{e}^{-ax}$.

Let us first normalize the eigenfunction ϕ:

$$\begin{aligned} 1 = A^2 \int_0^\infty x^2 \mathrm{e}^{-2ax}\, \mathrm{d}x &= \frac{A^2}{8a^3} \int_0^\infty y^2 \mathrm{e}^{-y}\, \mathrm{d}y \\ &= \frac{A^2}{4a^3}\,, \end{aligned}$$

where we have used the result $\int_0^\infty y^n \mathrm{e}^{-y}\, \mathrm{d}y = n!$. Therefore, $\phi = 2a^{3/2} x\mathrm{e}^{-2ax}$. Then

$$\begin{aligned} E &= \int_0^\infty \phi^* H \phi\, \mathrm{d}x \\ &= 4a^3 \int_0^\infty x\mathrm{e}^{-ax} \left(-\frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + Fx \right) x\mathrm{e}^{-ax}\, \mathrm{d}x \\ &= -\frac{2a^3\hbar^2}{m} \int_0^\infty x\mathrm{e}^{-ax} \left[-2a\mathrm{e}^{-ax} + a^2 x\mathrm{e}^{-ax} \right] \mathrm{d}x + 4a^3 F \int_0^\infty x^3 \mathrm{e}^{-2ax}\, \mathrm{d}x \\ &= \frac{\hbar^2 a^2}{m} \int_0^\infty y\mathrm{e}^{-y}\, \mathrm{d}y - \frac{\hbar^2 a^2}{4m} \int_0^\infty y^2 \mathrm{e}^{-y}\, \mathrm{d}y + \frac{F}{4a} \int_0^\infty y^3 \mathrm{e}^{-y}\, \mathrm{d}y \\ &= \frac{\hbar^2 a^2}{2m} + \frac{3F}{2a}\,. \end{aligned}$$

Next, $\partial E/\partial a = 0$ gives $a = (3Fm/(2\hbar^2))^{1/3}$. Then $E = (9/4)[2\hbar^2 F^2/(3m)]^{1/3}$.