LECTURE 7

No.
Date

Lets place our metal into a.c. electric field or what if irradiate the metal by light (e.g. laser).

We start w.th the same eqn.

The problem can be solved in the same Drude model where $\bar{f} = -e\bar{E}(t) = -e\,Re\left(E(\omega)\,e^{-i\omega t}\right)$

so $\frac{d\bar{p}}{dt} = -\bar{p}/\tau - e\bar{E}(t)$ we will seek a steady state solution i.e.

$$p(t) = Re\left(p(\omega)\,e^{-i\omega t}\right)$$

$$\frac{dp}{dt} = p(\omega)(-i\omega)e^{-i\omega t} = p(t)\cdot(-i\omega)$$

$+ i\omega\, p(\omega) = + \frac{p(\omega)}{\tau} + eE(\omega)$

$$\left(-i\omega + \frac{1}{\tau}\right)p(\omega) = eE(\omega)$$

$$p(\omega) = \frac{e}{\left(\frac{1}{\tau} - i\omega\right)}E(\omega)$$

Since $\bar{j} = -\frac{ne}{m}\bar{p} \Rightarrow$

$$j(t) = Re\, j(\omega)\, e^{-i\omega t}$$

$$j(\omega) = -\frac{ne}{m}p(\omega) = -\frac{ne^2}{m}\frac{E(\omega)}{\left(\frac{1}{\tau} - i\omega\right)}$$

Recall $j = \sigma E \Rightarrow \sigma(\omega) = -\frac{ne^2}{m}\cdot\frac{1}{\frac{1}{\tau} + i\omega} =$

$$= \sigma_0 / \left(\frac{1}{\tau} - i\omega\right)$$

$\sigma(\omega)$ is known as a.c. conductivity.

note $\omega \to 0 \to \sigma(\omega \to 0) = -\frac{ne^2}{m}\cdot\tau = \sigma_0$ = d.c. conductivity

Note: we ignored the magnetic field term. if we include it $\sim -\frac{e}{mc}\bar{p} \times \bar{H}$ ($\bar{H}$ ← magnetic field) $< E$ by about $\frac{v}{c}$! so at the 1st approximation ~ 0.

However the most important complication is that both E and H vary in space and time. But our $\sigma(\omega)$ was derived under the assumption that it is Spatially UNIFORM.

A few points are due:

1) the value of $\bar{j}$ at $\bar{r}$ is determined by E right after the very last collision, for any realistic material the mean free path as such that the last collision is few m.f.p. away from $\bar{r}$. So the fixed $E(r)$ should not vary too much. That is why we can write down:

$$\boxed{j(r,\omega) = \sigma(\omega)\, E(r,\omega), \quad \lambda \gg \text{m.f.p.}}$$

so this all valid if λ of the radiation $\Rightarrow$ Mean free path

m.f.p. of the electron.

Typical values for λs:

Red light (visible) ~ 800 nm

Soft X-rays (400 eV) ~ 1–2 nm

Hard X-rays (10 keV) ~ 1 Å

Now we can calculate the response function $\varepsilon(\omega)$ of a metal subjected to a.c. field.

$$\begin{cases} \nabla \bar{E} = 0 \\ \nabla H = 0 \\ \nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t} \\ \nabla \times H = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial E}{\partial t} \end{cases}$$

Recall $j(\omega) = \sigma(\omega) E(\omega)$ and we look for $E(r,t) = E(r)\, e^{-i\omega t}$

$$\nabla \times (\nabla \times E) = -\nabla^2 E = \frac{i\omega}{c}(\nabla \times H) =$$

$$= \frac{i\omega}{c}\left(\frac{4\pi\sigma}{c} E - \frac{i\omega}{c} E\right) \Rightarrow -\nabla^2 E = \frac{\omega^2}{c^2}\left(1 + \frac{4\pi}{\omega} i\sigma\right) E$$

$-\nabla^2 E = \frac{\omega^2}{c^2}\epsilon(\omega) E$ where $\epsilon(\omega) = 1 + \frac{4\pi i\sigma}{\omega}$

if ω is high, lets see what we get:

$$\sigma = \frac{\sigma_0}{\frac{1}{\tau} - i\omega} = \frac{\sigma_0}{1 - i\omega\tau} \underset{\omega\tau \gg 1}{=} -\frac{\sigma_0}{i\omega\tau} = -\frac{ne^2\tau}{m}\cdot\frac{1}{i\omega\tau}$$

then

$$\epsilon(\omega) \underset{\omega\tau \gg 1}{=} 1 - \frac{4\pi}{\omega}\frac{ne^2\tau}{m}\cdot\frac{1}{i\omega\tau} =$$

$$\boxed{\epsilon(\omega) = 1 - \frac{4\pi ne^2}{\omega^2 m} = 1 - \frac{\omega_p^2}{\omega^2}}$$ where $\boxed{\omega_p^2 = \frac{4\pi n e^2}{m}}$ PLASMA FREQUENCY

If $\frac{\omega_p^2}{\omega^2} < 1$ $\omega < \omega_p$ $\Rightarrow \epsilon(\omega) =$ = Real and positive

but if $\omega > \omega_p$

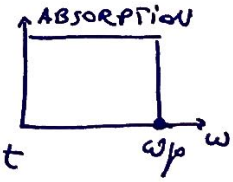
1) at $\omega = \omega_p$ $\epsilon = 0$! if $\omega < \omega_p$

ϵ is negative $\Rightarrow -\nabla^2 E = \frac{\omega^2}{c^2}\epsilon(\omega) E$

$$\nabla^2 E = \frac{\omega^2}{c^2}\epsilon(\omega) E$$

↑ negative

$E \sim e^{-\omega t}$



2) However if

$\epsilon > 0$ i.e. $\omega > \omega_p$

$E \sim e^{-i\omega t}$ oscillating behavior with no absorption.

Recall $\tau = \left(\frac{0.22}{\rho_\mu}\right)\left(\frac{r_s}{a_0}\right)^3 \cdot n^{-14}$ sec

$$\omega_p \cdot \tau = \frac{4\pi ne^2}{m}\cdot(\;) = 1.6\cdot 10^2 \left(\frac{r_s}{a_0}\right)^{3/2}\cdot\left(\frac{1}{\rho_\mu}\right)$$

$\omega_p \cdot \tau \gg 1$ 2 – 6 ↑ $\mu\Omega\cdot$cm

for any reasonable metal.

e.g. alkali metals like K, Na, Cs etc.

$$\nu_p = \frac{2\omega_p}{2\pi} = 11.4 \cdot \left(\frac{r_s}{a_0}\right)^{-3/2} \cdot 10^{15}\ \text{Hz} \Rightarrow$$

$$\lambda_p \sim 2 \cdot 10^3\ \text{Å}$$

② Perhaps among strange things is the presence of "CHARGE DENSITY oscillations" in a METAL.

From the continuity eqn.

$$\nabla \cdot j = -\partial\rho/\partial t \qquad \rho \sim \rho(\omega) e^{-i\omega t}$$

$$\begin{cases} \nabla \cdot j(\omega) = i\omega \rho(\omega) \\ \nabla \cdot E(\omega) = 4\pi \rho(\omega) \leftarrow \text{Gauss's law} \\ \text{and } j = \sigma(\omega) E \end{cases} \Rightarrow \nabla \cdot j = \sigma \nabla \cdot E$$

$$i\omega \rho(\omega) = \sigma(\omega) 4\pi \rho(\omega)$$

$$\rho(\omega)\left(1 - \frac{4\pi i \sigma(\omega)}{\omega}\right) = 0$$

$$\rho(\omega) \neq 0 \Rightarrow$$

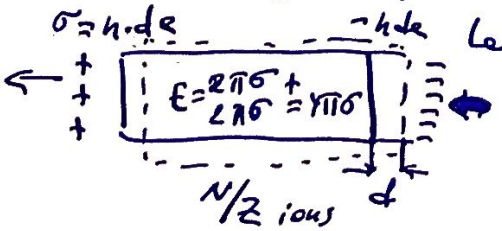
$$\boxed{1 + i\frac{4\pi\sigma(\omega)}{\omega} = 0}$$

← The condition the frequency should meet if the charge density wave to propogate.

HERE WE INTRODUCE A NEW CONCEPT - EXCITATIONS

The charge density wave before it gets quantized all also known as plasmons

A classic point of view:



Lets displace the e^- gas: here σ is not conductivity! it's surface charge density. For N electrons!

$$N m \ddot{d} = -Ne\,|4\pi\sigma| = -Ne(4\pi n d \cdot e) = -4\pi n e^2 N d$$

$$(q \cdot \bar{E}) \qquad \sim \omega^2$$

This leads to the oscillating motion.

L7

Thermal conductivity in a metal

In addition to the theory of conductivity Drude manages to explain what happens to a metal when we heat it up.

• Especially impressive how it works w.r.t the famous Wiedemann and Franz law;

$$\frac{\kappa}{\sigma} = \alpha T$$ where α is almost the same for all metals.

In other words

$$\frac{\kappa}{\sigma T} = \text{const}, \quad \text{e.g. Li } \frac{\kappa}{\sigma T} = 2.22 \cdot 10^{-8} \left[\text{Watt-}\Omega/\text{K}^2\right] \text{ at } 273\text{K}$$

Again as for the electrical conductivity lets assume that thermal transport is driven by electrons as well. Moreover, he suggested that since metals are more conductive than metals, the role of electrons must be very important compared to ions.

$Zn = 2.28\ldots$

$Pb = 2.64$

$Al = 2.14\ldots$

Let's assume we have a steady flow of heat

$$\vec{j}_T = -\kappa \nabla T$$

↑ thermal conductivity > 0 b/c thermal current is opposite to the thermal gradient.

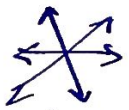
Lets assume we have a long thin rod.

$j_q = -\kappa \frac{dT}{dx}$: Now after each collision energies with the velocity $\sim$ to the local temperature,

so interestingly even if $\langle \bar{v} \rangle = 0$ the electron arriving from the hotter side will have higher energy than from the low T one.

So the net thermal flow is still perfectly maintained.

ε_1 $t_1 r_1$ → ε_2 $t_2 r_2$ $\quad \varepsilon(r_1) - \varepsilon(r_2) = \frac{d\varepsilon}{dT}(\nabla T)\cdot(r_2 - r_1) = \frac{C_v}{N}(\nabla T)(r_2 - r_1)$



higher T V.S. low T

if $r_2 - r_1$ along $x \Rightarrow \frac{C_v}{N}\nabla T\cdot(r_2 - r_1) = -\frac{C_v}{N}\frac{\partial T}{\partial x} v_x \cdot \tau$. For 1 electron

This is the amount of energy transfered from r_1 to r_2.

in homogenious system $\langle v_x^2 \rangle = \langle v_y \rangle^2 = \langle v_z \rangle^2$ $\quad \langle v^2 \rangle = v_x + v_y + v_z$

so $\langle v_x^2 \rangle = \frac{1}{3}\langle v^2 \rangle = 3\langle v_x^2 \rangle$

* see extra page

From kinetic theor of gases

$\kappa = \frac{1}{3} C_v \langle \bar{v} \rangle l$, where $l = \bar{v}\tau$ root mean square

$\bar{v}$ = is the average velocity

C_v is the heat capacity at const V, per volume

Lets take the ratio:

$$\frac{\kappa}{\sigma} = \frac{\frac{1}{3} C_v \langle v \rangle^2 \tau}{\frac{n e^2 \tau}{m}} =$$

$$C_v = \frac{3}{2} n k_B$$

$$= \frac{1/3 \cdot 3/2 \, n k_B \cdot v^2}{n e^2 / m} = \frac{1}{2}\frac{k_B \cdot \langle v^2 \rangle}{e^2/m}$$

we can calculate $\langle v \rangle^2$ from $m \cdot \frac{m\langle v^2 \rangle}{2} = \frac{3}{2} k_B T \cdot \kappa$

$$\Rightarrow \langle v^2 \rangle = \frac{3 k_B T}{m}$$

$$\frac{\kappa}{\sigma} = \frac{1}{2}\frac{k_B \cdot 3 k_B T}{e^2/m \cdot m} = \frac{3}{2}\frac{k_B^2}{e^2} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2 = 1.11 \cdot 10^{-8} \frac{\text{Watt}-\Omega}{K^2}$$

Which is great BUT only 1/2 of the experimental value!?

Note: in the original paper Drude made an error of 1/2 - so his result was perfect.

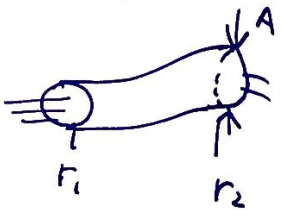
This is extra derivation for P6

(*)

derivation of κ in the kinetic theory of gases.

so we have determined, $E(r_2)-E(r_1) = \frac{C_V}{N}\frac{\partial T}{\partial x} v_x \tau$ go from $r_1 \to r_2$

so how many electrons within time δt



the number of electrons going through this tunnel:

$\langle A \cdot \underbrace{v_x \delta t}_{\text{volume } V \text{ of the tube}} \cdot n \rangle$ · carrier density $\frac{N}{V}$

So the energy transfer:

$$Q = -\frac{C_V}{N}\frac{\partial T}{\partial x} v_x \tau \cdot A \cdot n \delta t =$$

$$= -\frac{C_V}{N}\frac{\partial T}{\partial x} A n \langle v_x \rangle \delta t$$

since $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle \Rightarrow \langle v^2 \rangle = 3\langle v_x^2 \rangle$

so $$Q = -\frac{C_V}{N}\frac{\partial T}{\partial x} \tau A \cdot n \frac{\langle v^2 \rangle}{3} \delta t$$

The thermal current $j = \frac{Q}{A \delta t} = -\frac{C_V}{N} \tau n \frac{\langle v^2 \rangle}{3} \nabla \cdot T =$ ($n = N/V$)

for all 3 directions.

$$= -\frac{C_V}{V}\frac{\langle v^2 \rangle}{3} \tau \nabla T =$$

$= C_V$ ← heat capacity per volume.

often we can approximate $\sqrt{\langle v^2 \rangle} = v$ thus

$$j = -\frac{C_V \cdot v^2 \tau}{3} \nabla T$$, if we define the m.f.p $\ell = v\tau$

$$j = -\underbrace{\frac{C_V \cdot v \cdot \ell}{3}}_{\kappa} \nabla T \quad \text{so} \quad \boxed{\kappa = \frac{C_V v \ell}{3}}$$

Consider another interesting effect:

∇T flow of electrons will produce an internal electric field in the opposite direction to the flow of electrons.

This field is known as the thermoelectric field or the Seebeck effect.

$$\bar{E} = Q \nabla T$$

↑ thermopower.

To estimate the thermopower, we calculate electronic velocity at the point x (1D case)

mean $v_Q = \frac{1}{2}[v(x - v\tau) - v(x + v\tau)] = -\tau v \frac{dv}{dx} =$

$= -\tau \frac{d}{dx}\left(\frac{v^2}{2}\right)$

To generalize to 3D $\langle v_x^2 \rangle \to \frac{1}{3} v^2$ so that

$$v_Q = -\frac{\tau}{6} \frac{dv^2}{dT} \cdot \nabla T$$

the velocity Thermal due to the field gradient

, recall the average mean velocity due to the electric field $\bar{v}_E = -\frac{e\tau}{m} \cdot \bar{E}$

To have $v_Q + v_E = 0$

$$-\frac{\tau}{6} \frac{dv^2}{dT} \nabla T = -\frac{e\tau}{m} \cdot E \Rightarrow \bar{E} = +\frac{m}{e6} \frac{dv^2}{dT} \cdot \nabla T$$

or $Q = -\frac{m}{6e} \frac{dv^2}{dT} = -\frac{1}{3e} \left(\frac{d\, m v^2/2}{dT}\right)_{C_v} = -\frac{C_v}{3e} \cdot \frac{1}{n}$

Drude made another "mistake" stating

$C_v = \frac{3}{2} n k_B$ so $Q = -\frac{k_B}{2e} = -0.43 \cdot 10^{-4}\, V/K$

But for metals at RT $= -43\ \mu V/K$

the thermopower $Q(Li) \sim$ few μV or 100 times less!

This is a failure of the Drude theory.

We need QUANTUM MECHANICS to account for these kind of strange results!

THE END of L7