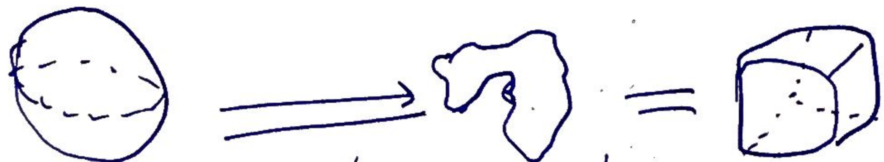
Lecture 17

Topology and insulators

Topology came from 3D real space but now moved to the Hilbert space

Def: if a manyfold M_1 can be <u>adiabatically</u> transformed into M_2, their topology is the same.



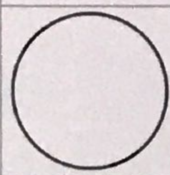
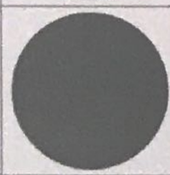
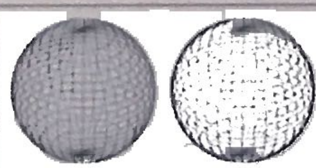
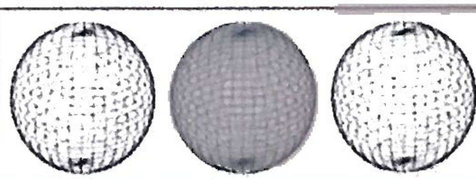
To distinguish between them we introduce an object called index (a topo index)

the same topo object = the same index

For 2D:
$$\chi_M = \frac{1}{2\pi} \oint K\, dS \quad \text{(the Euler characteristic)}$$

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V-E+F$
Tetrahedron		4	6	4	2
Hexahedron or cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

SEE WIKIPEDIA ON the EULER CHARACTERISTIC

Name	Image	Euler characteristic
Interval		1
Circle		0
Disk		1
Sphere		2
Torus (Product of two circles)		0
Double torus		−2
Triple torus		−4
Real projective plane		1
Möbius strip		0
Klein bottle		0
Two spheres (not connected) (Disjoint union of two spheres)		2 + 2 = 4
Three spheres (not connected) (Disjoint union of three spheres)		2 + 2 + 2 = 6

we locally fit the surface for to a particular circle and the $1/R$ is the local curvature κ

Among all the curvatures the largest and the smallest curvatures $\kappa_1 = \frac{1}{R_1}$ and $\kappa_2 = \frac{1}{R_2}$ are the <u>prrincipal curvatures</u>. The gaussian curvature

$$K = \frac{1}{\kappa_1} \cdot \frac{1}{\kappa_2}$$

For sphere $\kappa = \frac{1}{R_1} = \frac{1}{R_2}$

$$K = \frac{1}{R^2}$$

For a saddle point $\kappa_1 > 0$ and $\kappa < 0$

$\Rightarrow K = \kappa_1 \cdot \kappa_2 \leq 0$

Th: For any well-behaved 2D surface the $\oiint K\, dS = 2\pi \cdot n = \chi_M$ so it's quantized!
Also for the surfaces with the same topology χ_M are the same.

Th: For any <u>orientable</u> closed surface χ_M is always an <u>even</u> integer.

Orientable means we can distinguish two sides of the surface. If we cannot then χ_M is ODD (e.g. a Möbius strip)

— χ_M is a topological index only when the surface has no boundaries. Otherwise, it's not quantazided and not Topological.

Other topological properties:

① Topology and handles

χ_M is related to the genues of a surface (or manyfold)

$$\chi_M = 2(1-g).$$ where g is the number of handles of the object.

e.g. Sphere $g=0$ torus $g=1$

a coffee mug = a donut, or
3-handled cup = a pretzel = a triple torus

② χ_M and polyhedrons.

To define χ_M we can draw a grid of polyhedrons

$$\chi_M = \underset{\text{Corners}}{V} - \overset{\text{edges}}{E} + \underset{\text{faces}}{F}$$ e.g. since a sphere has $\chi_M = 2$ we know that $V-E+F=2$

③ Topology and hair vertex

If we draw a vector at each point of the surface. We get a vector field, which may have vortices. Now we define a verticity which is an integer number n, The total verticity $\chi_M = \sum n$.

Quantum Hall effect as an insulator

We start with the equation

$$E_n = \hbar\omega_c\left(n+\frac{1}{2}\right) \quad n=0,1,2\ldots$$

- We get equally spaced energy levels which are massively degenerate.
- The massive degeneracy can be seen in the ~~Landau~~ gauge $A = (-yB, 0, 0)$
- ~~Recal we ended up with~~

For sphere $\chi_M = 2 = \sum n = 2 \neq 0$
(~~top and~~ south and north poles)

The conclusion is, we cannot comb hair on the sphere without making the singularities (vertices).

Topological index for an insulator

The Berry curvature and the Chern number.

~~Recall~~ for an insulator we can use the Bloch waves to define a curvature in the 2D - k space. For Bloch waves

$\Psi_{n,k}(\bar{r}) = U_{n,k}(\bar{r}) e^{ik\bar{r}}$, we define the k-space curvature (Berry curvature) as

$$\mathcal{F}_n(k) = \iint_{\text{unit cell}} |\nabla_k U_{n,k}(\bar{r})|^* \times \nabla_k U_{n,k}(\bar{r}) d\bar{r} =$$

gradient ∇_k defines the vector in the k-space

$$= \epsilon_{ij} \iint_{U.C.} \left|\frac{\partial}{\partial k} U_{n,k}(\bar{r})\right|^* \frac{\partial}{\partial k} U_{n,k}(\bar{r})\, dr$$

ϵ_{ij} = Levi-Civita symbol $\quad \epsilon_{xx} = \epsilon_{yy} = 0$, $\epsilon_{xy} = -\epsilon_{yx} = 1$

$\boxed{\text{FROM the MATH p.o.v. the BERRY CURVATURE AND THE GAUSSIAN CURVATURE ARE THE SAME}}$

The total Berry curvature is the topological index!

The topological index is defined as

$$C_n = \frac{1}{2\pi} \oiint_{BZ} \mathcal{F}_n(k)\, d\bar{k} \equiv \text{the Chern number}$$

For each band n, we can define such a number C_n and for an insulator the total Chern #:

$$C = \sum_n C_n$$

$\uparrow$ over the filled bands

- The total chern number C_0, is the same as the number of chiral edge states.
  e.g. if $C=0$ we have a trivial insulator without edge states $\sigma_{xx} = \sigma_{xy} = 0$.
- If $C \neq 0$ we call such an insulator a TI or the Chern insulator.
  This insulator will have the edge states with $\boxed{\sigma_{xx} = 0}$ and $\boxed{\sigma_{xy} \neq 0 = C \frac{e^2}{\hbar}}$ for the Hall conductivity.
- Let me also claim without a proof.
  For a metal ~~or~~ an insulator, the Hall conductivity is the total Berry phase curvature summed over all occupied states. For metal we sum up over occupied (valence) and partially occupied bands (conduction bands):

$$\sigma_{xy} = \frac{e^2}{\hbar} \sum_{n,\,\text{valence band}} \left[\frac{1}{2\pi}\int_{BZ} d\bar{k}\, \mathcal{F}_n(\bar{k})\right] + \frac{e^2}{\hbar}\sum_{n,\,\text{conduction}} [\ldots]$$

- Few important points. For the gaussian curvature, the total K is only quantized if the surface has no boundaries
- For the Berry curvature is the same. if ~~the~~ we integrate over the whole BZ we will have a quantized Chern number.
- However, if we integrate over a part of BZ the C is non integer.
- For a metal we need to integrate only over the filled states or the Fermi sea and as such there is a boundary set by the Fermi surface. As such C is not quantized. That is why we have no quantized Hall conductivity for metals but we do have this for insulators.
- So by Chern # we define TI but not Topological metals.

OTHER TOPOLOGICAL INDICES.

If in addition to the C number, we demand a certain symmetry to be present e.g. time-reversal (TR) we can introduce different topo indices.

If any of these indees are non-zero the insulator is also a TI.
This kind of insulators also called the symmetry-protected insulators (SPTI), with the common properties:

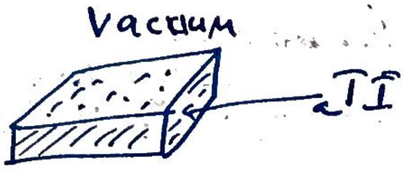
- One of the indees is non-zero.
- The bulk is an insulator, but the edge is a metalic state
- The edge state is different from a simple metal in d-1 dimensions. (e.g. 1/2 of the ordinary metal)
- The edge states may have some quantization effect.
- If the symmetry is broken the edge state disappear.

Note: if we assume no symmetry the only TI is the Chern insulator, which is defined in the even space dimensions, i.e. we can have QHE only in 2D but not in 3D.

Note: for SPTIs they can exis for both 2D & 3D if we preserve TR symm. (e.g. NO MAGNETISM)
In 1D we need a very special symmetry called the chiral symmetry to get a TI.

Q. Why TI have ~~any~~ metalic states at the edge?

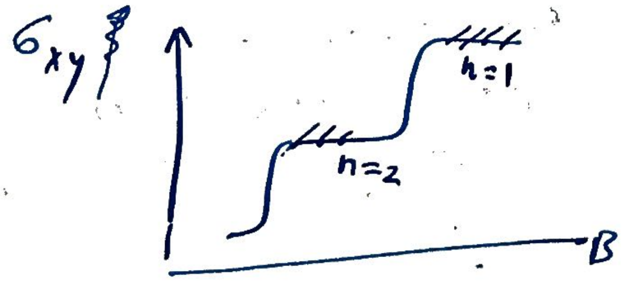
Consider



vacuum is an insulator (though trivial) with $C = 0$ inside the TI $C \neq 0$.

NB! Topology never changes in a smooth way! We cannot deform a sphere into a torus similarly we cannot transform a trivial or a band insulator into a TI, thus the insulating states need to be destroyed by closing a band gap, or we get a metal.

Q. Why there is a ~~p~~ metalic region between two plateaus?



different plateaus have different topological indeces n.
So the story as above to go from $n=1 \to n=2$ need to close a gap to destroy the topology. $\Rightarrow$ metal.

Q Why the Hall conductivity is so exact in a Chern TI?

Since the Hall conductivity is determined by topology of the wave function it's very robust and precise.

So ~~at~~ as long as any perturbation is not changing topology (or destroy protective symmetry) σ_{xy} will be the same for any sample.

~~So~~ In order to do this via some kind of perturbation we need to close a gap first (via doping for example) and only then we can change σ_{xy}.

So technically the error bar in σ_{xy} is 0. (well within how well we know $\hbar$ and e)

Q. So far you talked about non-interacting e^-. What if you turn e-e interactions?

For weakly interacting electrons the same connection between topology and Hall (the Berry connection) still remains. No proof here.