Solutions to the homework #9
(Due April 19)

① Griffiths 8.15

$$\int_0^1 F_2^{neutron}\,dx = 0.12 \qquad \int_0^1 F_2^{proton} = 0.18$$

Let's prove that $\int_0^1 x u(x)\,dx = 2\int_0^1 x\,d(x)\,dx$.

$$\begin{cases} F_2^{proton} = x\left\{\left(\frac{2}{3}\right)^2 u^{proton}(x) + \left(\frac{1}{3}\right)^2 d^{proton}(x)\right\} = x\left[\frac{4}{9}u^p(x) + \frac{1}{9}d^p(x)\right] \\ F_2^{neutron} = x\left\{\left(\frac{2}{3}\right)^2 u^{neutron}(x) + \left(\frac{1}{3}\right)^2 d^{neutron}(x)\right\} = x\left[\frac{4}{9}u^n(x) + \frac{1}{9}d^n(x)\right] \end{cases}$$

Due to $\int_0^1 F_2^{proton}\,dx = \frac{3}{2}\int_0^1 F_2^{neutron}\,dx$, we have

$$\int_0^1 x\left[\frac{4}{9}u^p + \frac{1}{9}d^p(x)\right]dx = \frac{3}{2}\int_0^1 x\left[\frac{4}{9}u^n(x) + \frac{1}{9}d^n(x)\right]dx$$

$$\Downarrow$$

$$\int_0^1 x\left[\frac{4}{9}u^p(x) + \frac{1}{9}d^p(x) - \frac{2}{3}u^n(x) - \frac{1}{6}d^n(x)\right]dx = 0$$

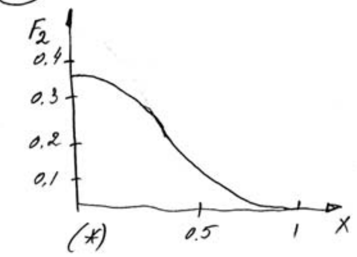
Due to ud symmetry $u^p(x) = d^n(x)$ and $u^n(x) = d^p(x)$.

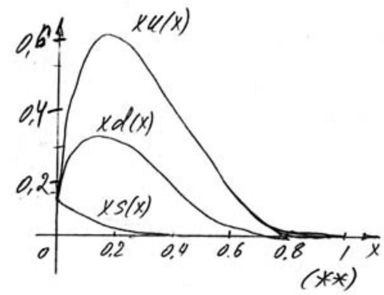
Finally we have

$$\int_0^1 x\left[\frac{5}{18}u^p(x) - \frac{10}{18}d^p(x)\right]dx = 0$$

$$\int_0^1 x\,u^p(x)\,dx = 2\int_0^1 x\,d^p(x)\,dx.$$

② Griffiths 8.17





$$F_2(x) = x\left[\left(\frac{2}{3}\right)^2 u(x) + \left(\frac{1}{3}\right)^2 d(x)\right]$$

The data on the figures are not compatible with the equation. They are inconsistant because the idea, that quarks inside the proton are free, is wrong.

In the area $x < 0.2$, in figure (**) $\frac{du(x)}{dx}$ and $\frac{dd(x)}{dx}$ are positive, but in figure (*) the slope $\frac{dF_2}{dx} < 0$.

③ Griffiths 10.2

Let's calculate the lifetime of the τ lepton.

$$\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau$$

$$\tau^- \to e^- \bar{\nu}_e \nu_\tau$$

$$\Gamma_\tau = \left(\frac{m_\tau g_w}{M_W}\right)^4 \frac{m_\tau c^2}{12\hbar(8\pi)^3} \Rightarrow \text{The lifetime} \quad \tau_{\tau(1)} = \frac{1}{\Gamma_\tau} = \left(\frac{M_W}{m_\tau g_w}\right)^4 \frac{12\hbar(8\pi)^3}{m_\tau c^2}$$

Substituting all values $g_w = 0.66$

$$M_W = 80.4 \ \frac{GeV}{c^2}$$

$$m_\tau = 1.784 \ \frac{GeV}{c^2}$$

Finally $\tau_{\tau(1)} = 1.528 \times 10^{-12}\,s$

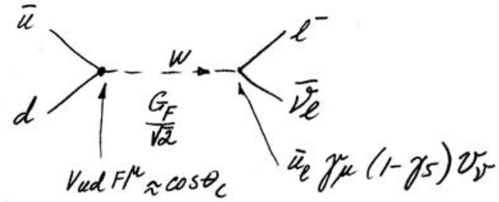
Taking into account that we have two decay modes we double the answer:

$$\tau_\tau \approx 3 \times 10^{-12}\,s$$

From Particle Data Group list $\tau_\tau = 2.9 \times 10^{-13}\,s$, which is very close to our calculations.

(4) Griffith 10.10

$\pi^- \to \ell^- \bar{\nu}_\ell$



$$\mathcal{M} = \frac{G_F}{\sqrt{2}} c_1 f_\pi (p^\mu + k^\mu) [\bar{u}(p) \gamma_\mu (1-\gamma_5) v(k)] =$$

$$= \frac{G_F}{\sqrt{2}} c_1 f_\pi [\bar{u}(p) \not{p} (1-\gamma_5) v(k) + \bar{u}(p)(1+\gamma_5) \not{k} v(k)]$$

Using Dirac equation:

$$\begin{cases} (\not{k} + m_\nu) v(k) = 0 \\ \bar{u}(p)(\not{p} - m_\ell) = 0 \end{cases} \Rightarrow \quad \begin{matrix} \not{k}\, v(k) = -m_\nu = 0 \\ \bar{u}(p) \not{p} = \bar{u}(p)\, m_e \end{matrix}$$

we have:

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} c_1 f_\pi m_e \bar{u}(p)(1-\gamma^5) v(k)$$

$$|\mathcal{M}|^2 = \frac{G_F^2}{2} c_1^2 f_\pi^2 m_e^2 \sum_{spins} [\bar{u}(p)(1-\gamma^5) v(k)][\bar{v}(k)(1+\gamma^5) u(p)] =$$

$$= 4 G_F^2 c_1 f_\pi^2 m^2 (p \cdot k)$$

Due to $E_\nu = |\vec{p}_f| \Rightarrow E_\ell + E_\nu = m_\pi = (m_e^2 + E_\nu^2)^{1/2} + E_\nu = m_\pi \Rightarrow$

$$\Rightarrow E_\nu = \frac{m_\pi^2 - m_e^2}{2 m_\pi}$$

Hence $p \cdot k = E_\ell E_\nu - |\vec{p}_f|^2 \cos\theta = E_\ell E_\nu - E_\nu^2 \cos\theta \, ; =$

~~$m_e f (E_\ell - E_\nu \cos\theta)$~~

Also $\Gamma = v\sigma |\psi(0)|^2$

Combining all together we have $f_\pi^2 = \frac{4\hbar^3 m}{m_\pi^2 c} \cos^2\theta_c |\psi(0)|^2$

(5) Griffiths, 10.12

$$\frac{\Gamma(K^- \to e^- + \bar{\nu}_e)}{\Gamma(K^- \to \mu^- + \bar{\nu}_\mu)} = \frac{m_e^2 (m_{K^-}^2 - m_e^2)^2}{m_\mu^2 (m_{K^-}^2 - m_\mu^2)^2}$$

Substituting $m_e = 0.511\ MeV/c^2$, $m_\mu = 105.66\ MeV/c^2$, $m_{K^-} = 493.67\ MeV/c^2$ we have $\frac{\Gamma(K^- \to e^- + \bar{\nu}_e)}{\Gamma(K^- \to \mu^- + \bar{\nu}_\mu)} = \boxed{2.57\times10^{-5}}$

Let's estimate f_K using information that $\tau_{K^-} = 1.2\times10^{-8}s$

$$\Gamma(K^- \to \mu^- + \bar{\nu}_\mu) = 0.64\,\Gamma_{tot} = \frac{0.64}{\tau_{K^-}} = \frac{0.64}{1.2\times10^{-8}s} = 5.33\times10^{7} s^{-1}$$

$$\Gamma(K^- \to \mu^- + \bar{\nu}_\mu) = \frac{f_K^2}{\pi\hbar m_{K^-}^3}\left(\frac{g_w}{4M_W}\right)^4 m_\mu^2 (m_K^2 - m_\mu^2)^2 \Rightarrow$$

$$\Rightarrow f_K = \frac{\sqrt{\Gamma(K^- \to \mu^- + \bar{\nu}_\mu)\,\pi\hbar\, m_{K^-}^3}\,(4M_W)^2}{g_w^2\, m_\mu\, (m_K^2 - m_\mu^2)} = \boxed{35.22\ MeV}$$

From Particle Data Group

$f_K \simeq 160\ MeV.$