Solutions to the homework #8 (Due April 12)

① Griffiths 7.2

Let's show that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ for $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

By definition $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$

$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$; $\sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad g^{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$

Hence, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{bmatrix} =$

$= \begin{bmatrix} -\sigma^\mu\sigma^\nu & 0 \\ 0 & -\sigma^\mu\sigma^\nu \end{bmatrix} + \begin{bmatrix} -\sigma^\nu\sigma^\mu & 0 \\ 0 & -\sigma^\mu\sigma^\nu \end{bmatrix} = 0$ for $j \neq k$,

where $\sigma^\mu\sigma^\nu = -\sigma^\nu\sigma^\mu$

For γ^0 : $\{\gamma^0 \gamma^0\} = 2\gamma^0\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

For γ^μ : $\{\gamma^\mu \gamma^\mu\} = 2\gamma^\mu\gamma^\mu = 2\begin{bmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{bmatrix} = 2\begin{bmatrix} -\sigma^\mu\sigma^\mu & 0 \\ 0 & -\sigma^\mu\sigma^\mu \end{bmatrix} = 2$

Finally $\boxed{\{\gamma^\mu \gamma^\nu\} = 2g^{\mu\nu}}$

(2) Griffith 7.6

In general case

$$u^{(1)} = N\begin{bmatrix} 1 \\ 0 \\ \frac{c(p_z)}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{bmatrix} \quad u^{(2)} = N\begin{bmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{c(-p_z)}{E+mc^2} \end{bmatrix} \quad u^{(3)} = N\begin{bmatrix} \frac{c(p_z)}{E-mc^2} \\ \frac{c(p_x+ip_y)}{E-mc^2} \\ 1 \\ 0 \end{bmatrix} \quad u^{(4)} = N\begin{bmatrix} \frac{c(p_x-ip_y)}{E-mc^2} \\ \frac{c(-p_z)}{E-mc^2} \\ 0 \\ 1 \end{bmatrix}$$

If the z axis points along the direction of motion, we have:

$$u^{(1)} = \sqrt{\frac{E+mc^2}{c}}\begin{bmatrix} 1 \\ 0 \\ \frac{c(p_z)}{E+mc^2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{E+mc^2}{c}} \\ 0 \\ \sqrt{\frac{(E+mc^2)c^2}{c(E+mc^2)^2}}\, p_z \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{(E+mc^2)/c} \\ 0 \\ \sqrt{\frac{c}{E+mc^2}}\,\frac{\sqrt{E^2-m^2c^4}}{c} \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \sqrt{(E+mc^2)/c} \\ 0 \\ \frac{\sqrt{(E-mc^2)(E+mc^2)}}{\sqrt{(E+mc^2)c}} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{(E+mc^2)/c} \\ 0 \\ \sqrt{(E-mc^2)/c} \\ 0 \end{bmatrix}$$

Doing the same procedure for $u^{(2)}$; $u^{(3)}$ and $u^{(4)}$, we have:

$$u^{(2)} = \begin{pmatrix} 0 \\ \sqrt{(E+mc^2)/c} \\ 0 \\ -\sqrt{(E-mc^2)/c} \end{pmatrix}; \quad u^{(3)} = \begin{pmatrix} \frac{E+mc^2}{c\sqrt{(E-mc^2)/c}} \\ 0 \\ \sqrt{(E+mc^2)/c} \\ 0 \end{pmatrix}; \quad u^{(4)} = \begin{pmatrix} 0 \\ -\frac{(E+mc^2)/c}{\sqrt{(E-mc^2)/c}} \\ 0 \\ \sqrt{(E+mc^2)/c} \end{pmatrix}$$

Let's show that they are eigenspinors of S_z.

$S_3 = \frac{\hbar}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$, where $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, hence $S_3 = \frac{\hbar}{2}\begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}$

Finally

$S_3 u^{(1)} = \frac{\hbar}{2} u^{(1)} \Rightarrow u^{(1)}$ is an eigenspinor of S_3 with eigenvalue $\boxed{\frac{\hbar}{2}}$

$S_3 u^{(2)} = -\frac{\hbar}{2} u^{(2)} \Rightarrow u^{(2)}$ is an eigenspinor of S_3 with eigenvalue $\boxed{-\frac{\hbar}{2}}$

$S_3 u^{(3)} = \frac{\hbar}{2} u^{(3)} \Rightarrow u^{(3)}$ is an eigenspinor of S_3 with eigenvalue $\boxed{\frac{\hbar}{2}}$

$S_3 u^{(4)} = -\frac{\hbar}{2} u^{(4)} \Rightarrow u^{(4)}$ is an eigenspinor of S_3 with eigenvalue $\boxed{-\frac{\hbar}{2}}$

③ Griffiths, 7.13

Let's show that adjoint spinors $\bar{u}^{(1,2)}$ and $\bar{v}^{(1,2)}$ satisfy the equations

$$\bar{u}(\gamma^\mu p_\mu - mc) = 0 \qquad \bar{v}(\gamma^\mu p_\mu + mc) = 0$$

Due to $(\gamma^0)^+ = \gamma^0$ and for any matrixes A and B $(AB)^+ = B^+A^+$, let's conjugate the following equations:

$$\begin{cases} \left[(\gamma^\mu p_\mu - mc)u\right]^+ = 0 \\ \left[(\gamma^\mu p_\mu + mc)v\right]^+ = 0 \end{cases} \Rightarrow \begin{cases} u^+(\gamma^\mu p_\mu - mc)^+ = 0 \\ v^+(\gamma^\mu p_\mu + mc)^+ = 0 \end{cases}$$

Let's multiply the equations by γ^0 from the right:

$$\begin{cases} u^+(\gamma^\mu p_\mu - mc)^+\gamma_0 = 0 \\ v^+(\gamma^\mu p_\mu + mc)^+\gamma_0 = 0 \end{cases} \Rightarrow \begin{cases} u^+(\gamma^\mu p_\mu - mc)^+\gamma_0^+ = 0 \\ v^+(\gamma^\mu p_\mu + mc)^+\gamma_0^+ = 0 \end{cases} \Rightarrow$$

But $(\gamma^\mu p_\mu - mc)^+\gamma_0^+ = \gamma^0(\gamma^\mu p_\mu - mc)$, and $\begin{cases} u^+\gamma^0 = \bar{u} \\ v^+\gamma^0 = \bar{v} \end{cases}$, hence

we have

$$\begin{cases} u^+\gamma^0(\gamma^\mu p_\mu - mc) = 0 \\ v^+\gamma_0(\gamma^\mu p_\mu + mc) = 0 \end{cases} \Rightarrow \boxed{\begin{array}{c} \bar{u}(\gamma^\mu p_\mu - mc) = 0 \\ \bar{v}(\gamma^\mu p_\mu - mc) = 0 \end{array}}$$

④ Griffiths 7.44

(a) For pair annihilation $e^+ + e^- \to \gamma + \gamma$ let's find the amplitude $\mathcal{M}$.

$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$.

$$-i\mathcal{M}_1 = (2\pi)^4 \int [\bar{v}(2)(ig_e \gamma^\mu)(i(\gamma^\mu q_u + mc)) / (q^2 - mc^2))(ig_e \gamma^\nu)\, u(1)] \cdot$$
$$\cdot\, \delta^4(p_1 - p_3 - q)\, \delta^4(q + p_2 - p_4)\, \delta^4 q =$$
$$= \frac{i^3 g_e^2}{(q^2 - mc^2)} \bar{v}(2)\gamma^\mu(\gamma^\mu q_u + mc)\gamma_\mu^\nu\, u(1) = -\frac{ig_e^2}{(p_1 - p_3)^2 - mc^2} \bar{v}(2)\gamma^\mu(p_1 - p_3 - mc^2) \times$$
$$\times\, \gamma_\mu^\nu\, u(1);$$

Hence $\mathcal{M}_1 = \dfrac{g_e^2}{(p_1 - p_3)^2 - mc^2} \bar{v}(2)\gamma^\mu(p_1 - p_3 - mc^2)\gamma^\mu_\nu\, u(1)$

We have the same procedeure for $\mathcal{M}_2$, where $q = p_1 - p_4$:

$$\mathcal{M}_2 = \frac{g_e^2}{(p_1 - p_3)^2 - mc^2} \bar{v}(2)\gamma^\mu(p_1 - p_4 - mc^2)\gamma_\mu^\nu\, u(1).$$

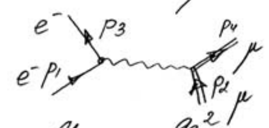
Finally: $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 =$

$$= g_e^2 \left[\frac{1}{(p_1 - p_4)^2 - mc^2} \bar{v}(2)\gamma^\mu(p_1 - p_4 - mc^2)\gamma_\mu^\nu\, u(1) + \frac{1}{(p_1 - p_3)^2 - mc^2} \bar{v}(2)\gamma^\mu(p_1 - p_3 - mc^2)\gamma_\mu^\nu\, u(1)\right]$$

(b) Let's determine $\langle |\mathcal{M}|^2 \rangle$, considering that $m_e = 0$, $m_\gamma = 0$.

$$\mathcal{M} = g_e^2 \left[\frac{1}{(p_1 - p_4)^2} \bar{v}(2)\gamma^\mu(p_1 - p_4)\gamma_\mu^\nu\, u(1) + \frac{1}{(p_1 - p_3)^2} \bar{v}(2)\gamma^\mu(p_1 - p_2)\gamma_\mu^\nu\, (1)\right.$$

(5) For $e\mu$ scattering we have:



$e+\mu \to e+\mu$ (Mott scattering ($M >> m$) $\Rightarrow$ Rutherford scattering ($v << c$)

$$\mathcal{M} = -\frac{g_e^2}{(p_1-p_3)^2}\left[\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)\right]\left[\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)\right]$$

The differential cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\overset{1}{\overset{\shortparallel}{S}}|\mathcal{M}|^2}{(E_1+E_2)^2}\frac{|p_f|}{|p_i|} = \left(\frac{\hbar c}{8\pi(E_1+E_2)}\right)^2 |\mathcal{M}|^2$$

$$(p_1-p_3)^2 = p^2\sin^2\theta + p(1-\cos\theta)^2 = p^2(\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta) = 4p^2\sin^2\frac{\theta}{2}$$

Hence,

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \left(\frac{\hbar c}{8\pi\left[\sqrt{p^2c^2+m^2c^4} + \sqrt{p^2c^2+M^2c^4}\right)}\right) \frac{g_e^2}{4p^2\sin^2\frac{\theta}{2}}$$