Solutions to the homework #6.
(Due March 10^{th})

① Let's prove that $\gamma^2 - \eta^2 = 1$.

Due to $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$ and $\eta = \beta\gamma \Rightarrow \gamma^2 = \frac{1}{1-\beta^2} = \frac{1}{1-\left(\frac{\eta}{\gamma}\right)^2} \Rightarrow$

$\Rightarrow \boxed{\gamma^2 - \eta^2 = 1}$

③ Griffiths 3.17

For the decay $A \to B + C$ $\quad E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A} c^2$; $|p_B| = |p_C| = \frac{\sqrt{\lambda(m_A^2, m_B^2, m_C^2)}}{2m_A} c$

(a) $\pi^- \to \mu^- + \bar{\nu}_\mu$

$$E_{\mu^-} = \frac{m_{\pi^-}^2 + m_{\mu^-}^2 - m_{\bar{\nu}_\mu}^2}{2m_{\pi^-}} c^2 = \frac{(139.56)^2 + (105.66)^2 - 0^2}{2 \times 139.56} \simeq \boxed{109.78\ MeV}$$

$$E_{\bar{\nu}_\mu} = \frac{m_{\pi^-}^2 + m_{\bar{\nu}_\mu}^2 - m_{\mu^-}^2}{2m_{\pi^-}} c^2 = \frac{(139.56)^2 + 0^2 - (105.66)^2}{2 \times 139.56} \simeq \boxed{29.79\ MeV}$$

(b) $\pi^0 \to \gamma + \gamma$

$$E_\gamma = \frac{m_{\pi^0}^2 + m_\gamma^2 - m_\gamma^2}{2m_{\pi^0}} c^2 = \frac{m_{\pi^0}}{2} = \frac{134.96}{2} = \boxed{67.48\ MeV}$$

(c) $K^+ \to \pi^+ + \pi^0$

$$E_{\pi^+} = \frac{m_{\pi^+}^2 + m_{K^+}^2 - m_{\pi^0}^2}{2m_{K^+}} c^2 = \frac{(139.57)^2 + (493.67)^2 - (134.96)^2}{2 \times 493.67} \simeq \boxed{248.12\ MeV}$$

$$E_{\pi^0} = \frac{m_{K^+}^2 + m_{\pi^0}^2 - m_{\pi^+}^2}{2m_{K^+}} c^2 = \frac{(493.67)^2 + (134.96)^2 - (139.57)^2}{2 \times 493.67} \simeq \boxed{245.56\ MeV}$$

(d) $\Lambda \to p + \pi^-$

$$E_p = \frac{m_\Lambda^2 + m_p^2 - m_{\pi^-}^2}{2m_\Lambda} c^2 = \frac{(1115.6)^2 + (938.28)^2 - (139.57)^2}{2 \times 1115.6} \simeq \boxed{943.64\ MeV}$$

$$E_{\pi^-} = \frac{m_\Lambda^2 + m_{\pi^-}^2 - m_p^2}{2m_\Lambda} c^2 = \frac{(1115.6)^2 + (139.57)^2 - (938.28)^2}{2 \times 1115.6} \simeq \boxed{171.96\ MeV}$$

(e) $\Omega^- \to \Lambda + K^-$

$$E_\Lambda = \frac{m_\Omega^2 + m_\Lambda^2 - m_K^2}{2m_\Omega} c^2 = \frac{(1672)^2 + (1115.6)^2 - (493.7)^2}{2 \times 1672} \simeq \boxed{1135.3\ \text{MeV}}$$

$$E_{K^-} = \frac{m_{\Omega^-}^2 + m_{K^-}^2 - m_\Lambda^2}{2m_{\Omega^-}} c^2 = \frac{(1672)^2 + (493.7)^2 - (1115.6)^2}{2 \times 1672} \simeq \boxed{536.7\ \text{MeV}}$$

4) Griffiths, 3.19

A particle A at rest : $A \to B + C + D + \ldots$

(a) Let's find $E_{B\,min}$ and $E_{B\,max}$.

Due to $E_B = \dfrac{m_A^2 + m_B^2 - (m_C^2 + m_D^2 + \ldots)}{2m_A} c^2$ the maximum energy is when particles $C, D, \ldots$ are at rest.

Hence

$$\boxed{E_{B_{max}} = \frac{m_A^2 + m_B^2 - (m_C^2 + m_D^2 + \ldots)}{2m_A} c^2} \qquad \boxed{E_{min} = m_B c^2}$$

(b) $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$

The minimum energy of the electron $E_{e_{min}} = m_e c^2 = \boxed{0.511\ \text{MeV}}$

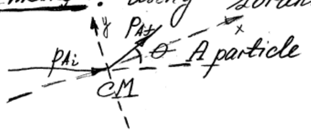
The maximum energy of the electron $E_{e\,max} = \dfrac{m_\mu^2 + m_e^2}{2m_\mu} = \boxed{52.83\ \text{MeV}}$

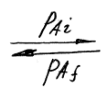
5) Griffiths, 3.21

$A + B \to A + B$ elastic scattering. Let's find energy and velocity in Breit frame.

$\vec{P}_{Ai} = -\vec{P}_{Af}$, $E_{A_i^f} = E$, θ. I know two solutions for the problem :

1st method: Using Lorentz transformation, for particle A : $P_{CM_x} = p \cos\frac{\theta}{2}$, $P_{CM_y} = p \sin\theta/2$





Breit frame

$$\begin{cases} P_{Breit_x} = \gamma(P_{CM_x} - \beta E) = 0 \Rightarrow \beta = \frac{P_{CM_x}}{E} \\ P_{Breit_y} = P_{CM_y} \end{cases}$$

$$\Rightarrow \begin{cases} \beta = \dfrac{P_{CM_x}}{E} = \dfrac{p \cos\frac{\theta}{2}}{E} \\ P_A = P_{Breit_y} = \boxed{p \sin\theta/2} \end{cases}$$

The energy $E_{Breit_A}^2 = P_A^2 c^2 + m_A^2 c^4 = p^2 \sin^2\left(\frac{\theta}{2}\right) c^2 + m_A^2 c^4 =$

$$= \boxed{\frac{1}{2}\left[E^2(1 - \cos\theta) + m^2 c^4 (1 + \cos\theta)\right]}$$

2nd method: Due to multiplication of two four-vectors is invariant,

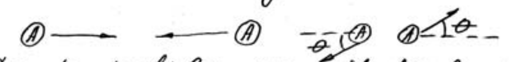
$p_{Ai_{CM}} \cdot p_{Af_{CM}} = p_{Ai_{Br}} \cdot p_{Af_{Br}}$; <u>CM</u>: $p_{Ai} = (p, E/c)$, $p_{Af} = (p\cos\theta, E/c)$; <u>Breit:</u> $p_{Ai} = (p_A, E_A/c)$, $p_{Af} = (-p_A, E_A/c)$

Hence $\begin{cases} E^2 - p^2\cos\theta = E_A^2 + p_A^2 \\ E^2 - p^2c^2 = m^2c^4 = E_A^2 - p_A^2c^2 \end{cases} \Rightarrow$

$$\Rightarrow E_A^2 = \frac{1}{2}\left[E^2 - (E^2 - m^2c^4)\cos\theta + m^2c^4\right] = \boxed{\frac{1}{2}\left[E^2(1-\cos\theta) + m^2c^4(1+\cos\theta)\right]}$$

(6) <u>Griffiths 3.23</u>

For elastic scattering $A+A \to A+A$, Mandelstam variables s, t, u?



Due to particles are identical, the momentums are p, and energies are the same. Hence:

$p_{A_1 i} = (\vec{p}, \frac{E}{c})$; $p_{A_2 i} = (-\vec{p}, E/c)$; $p_{A_1 f} = (p\cos\theta, p\sin\theta, 0, E/c)$

$p_{A_2 f} = (-p\cos\theta, -p\sin\theta, 0, E/c)$

$$p_i = p_{A_1 i} + p_{A_2 i} = (0, \frac{2E}{c}) \Rightarrow p_i^2 = \frac{4E^2}{c^2} \Rightarrow s = \frac{p_i^2}{c^2} = \boxed{\frac{4(p^2+m^2c^2)}{c^2}}$$

$$t = \frac{(p_{A_i} - p_{Af})^2}{c^2} = \frac{-p^2(1-\cos\theta)^2 - p^2\sin^2\theta}{c^2} = \frac{-2p^2 + 2\cos\theta \cdot p^2}{c^2} = \boxed{\frac{-2p^2(1-\cos\theta)}{c^2}}$$

$$u = \frac{(p_{A_1 i} - p_{A_2 f})^2}{c^2} = \boxed{-\frac{2p^2(1+\cos\theta)}{c^2}}$$

(7) <u>Griffiths 4.11</u>

$\Delta^{++} \to p + \pi^+$. Let's find ℓ in the final state.

In the final state the total spin is $S_{tot} = S_p + S_\pi = 1/2 + 0 = 1/2$

In a Δ^{++} rest frame the total angular momentum $J = S_{\Delta^{++}} = 3/2$.

From conservation of the angular momentum $J_i = J_f \Rightarrow$

$$\vec{J}_f = \vec{S} + \vec{L} = \frac{3}{2} \Rightarrow \begin{cases} -S + L = 3/2 \\ S + L = 3/2 \end{cases} \Rightarrow \boxed{\begin{matrix} L = 1 \\ L = 2 \end{matrix}}$$

⑧ Griffiths, 4.21

(a) $e^{i\frac{\pi}{2}\sigma_z} = I + i\frac{\pi}{2}\sigma_z + \frac{1}{2} i^2 \left(\frac{\pi}{2}\right)^2 \sigma_z^2 + \frac{1}{3!} i^3 a^3 \sigma_z^3 + \frac{1}{4!} i^4 \left(\frac{\pi}{2}\right)^4 \sigma_z^4 + \dots =$

$= I - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 \sigma_z^2 + \frac{1}{4!}\left(\frac{\pi}{2}\right)^4 \sigma_z^4 + \dots + i\sigma_z\left(\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^3 \frac{\sigma_z^2}{3!} + \frac{1}{5!}\left(\frac{\pi}{2}\right)^5 \sigma_z^4 + \dots\right) =$

$= 1 - \left(\frac{\pi}{2}\right)^2 \frac{1}{2} + \frac{1}{4!}\left(\frac{\pi}{2}\right)^4 + \dots + i\sigma_z\left(\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^3 \frac{1}{3!} + \dots\right) = \cos\frac{\pi}{2} + i\sigma_z \sin\frac{\pi}{2} = \boxed{i\sigma_z}$

(b) We want matrix U that $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} Ab + Ba \\ Cb + Da \end{pmatrix} \Rightarrow \begin{cases} A=0 \\ B=1 \\ C=1 \\ D=0 \end{cases}$

The matrix is $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence for spin up $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and spin down $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have: $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) Let's show that $U(\theta) = \cos\frac{\theta}{2} - i(\hat{\theta}\cdot\hat{\sigma})\sin\frac{\theta}{2}$.

Due to $U(\theta) = e^{-\frac{i\hat{\theta}\hat{\sigma}}{2}} = \cos\left(\frac{\hat{\theta}\hat{\sigma}}{2}\right) - i\sin\left(\frac{\hat{\theta}\hat{\sigma}}{2}\right)$, for $\vec{a}\times\vec{b}$: $(\hat{\sigma}\vec{a})(\hat{\sigma}\vec{b}) =$
$= \vec{a}\cdot\vec{b} + i\hat{\sigma}(\vec{a}\times\vec{b})$. Taking $\hat{\theta} = \vec{a} = \vec{b} \Rightarrow (\hat{\theta}\cdot\hat{\sigma})^2 = \theta^2 \Rightarrow \hat{\theta}\hat{\sigma} = \pm\theta$

Hence $\boxed{U(\theta) = \cos\frac{\theta}{2} + i(\hat{\theta}\hat{\sigma})\sin\frac{\theta}{2}}$

⑨ Griffiths, 4.24

Let's find isospin assignments $|I, I_3\rangle$ for Ω^-, Σ^+, Ξ^0, p^+, η, $\bar{K}^0$

Due to $I_u = 1/2$ $I_{3u} = +1/2$; $I_d = 1/2$ $I_{3d} = -1/2$; $I_s = 0$; $I_{3s} = 0$

$\Omega^- = sss$ $|I, I_3\rangle = |0;0\rangle$

$\Sigma^+ = uus$ $|I, I_3\rangle = |1;1\rangle$

$\Xi^0 = uss$ $|I, I_3\rangle = |1/2;1/2\rangle$

$p = uud$ $|I, I_3\rangle = |1/2;1/2\rangle$

$\eta = \frac{1}{\sqrt{6}}(d\bar{d} + u\bar{u} - 2s\bar{s})$ $|I, I_3\rangle = |0;0\rangle$

$\bar{K}^0 = s\bar{d}$ $|I, I_3\rangle = |1/2;1/2\rangle$

⑩ Griffiths, 4.28

ⓐ In pion-nucleon scattering $\pi N - \pi N$:

(a) $\pi^+ + p \to \pi^+ + p$	M_3	Due to the initial and final states are the same.
(b) $\pi^0 + p \to \pi^0 + p$	$\frac{2}{3}M_3 - \frac{1}{3}M_1$	
(c) $\pi^- p \to \pi^- p$	$\frac{1}{3}M_3 + 2/3\,M_1$	
(d) $\pi^+ + n \to \pi^+ + n$	$1/3\,M_3 + 2/3\,M_1$	
(e) $\pi^0 n \to \pi^0 + n$	$2/3\,M_3 + 1/3\,M_1$	

(f) $\pi^- + n \to \pi^- + n \qquad M = M_3$

In the next reactions there is a charge exchange, i.e. $|\psi_i\rangle \neq |\psi_f\rangle$

(g) $\pi^+ + n \to \pi^0 + p$

Due to $M = \langle \psi_f | \hat{H} | \psi_i \rangle$

$$\left.\begin{aligned} |\psi_i\rangle &= \frac{1}{\sqrt{3}} |\tfrac{3}{2}\, \tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{2}}\, |\tfrac{1}{2}\, \tfrac{1}{2}\rangle \\ |\psi_f\rangle &= \sqrt{\tfrac{2}{3}}\, |\tfrac{3}{2}\, \tfrac{1}{3}\rangle - \frac{1}{\sqrt{3}} |\tfrac{1}{2}\, \tfrac{1}{2}\rangle \end{aligned}\right\} \Rightarrow M = \frac{\sqrt{2}}{3} M_3 - \frac{1}{\sqrt{2}} M_1$$

(h) $\pi^0 + p \to \pi^+ + n$

$$\left.\begin{aligned} |\psi_i\rangle &= \sqrt{\tfrac{2}{3}}\, |\tfrac{3}{2}\, \tfrac{1}{2}\rangle - \frac{1}{\sqrt{3}} |\tfrac{1}{2}\, \tfrac{1}{2}\rangle \\ |\psi_f\rangle &= \frac{1}{\sqrt{3}} |\tfrac{3}{2}\, \tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{2}}\, |\tfrac{1}{2}\, \tfrac{1}{2}\rangle \end{aligned}\right\} \Rightarrow M = \sqrt{\tfrac{2}{9}}\, M_3 - \sqrt{\tfrac{1}{2}}\, M_1$$

(i) $\pi^0 + n \to \pi^- + p$

$$\left.\begin{aligned} |\psi_i\rangle &= \sqrt{\tfrac{2}{3}}\, |\tfrac{3}{2} -\tfrac{1}{2}\rangle + \frac{1}{\sqrt{3}} |\tfrac{1}{2} -\tfrac{1}{2}\rangle \\ |\psi_f\rangle &= \sqrt{\tfrac{1}{3}}\, |\tfrac{3}{2} -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |\tfrac{1}{2} -\tfrac{1}{2}\rangle \end{aligned}\right\} \Rightarrow M = \sqrt{\tfrac{2}{9}}\, M_3 - \sqrt{\tfrac{2}{9}}\, M_1$$

(j) $\pi^- + p \to \pi^0 + n$

$$\left.\begin{aligned} |\psi_i\rangle &= \sqrt{\tfrac{1}{3}}\, |\tfrac{3}{2} -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |\tfrac{1}{2} -\tfrac{1}{2}\rangle \\ |\psi_f\rangle &= \sqrt{\tfrac{2}{3}}\, |\tfrac{3}{2} -\tfrac{1}{2}\rangle + \frac{1}{\sqrt{3}} |\tfrac{1}{2} -\tfrac{1}{2}\rangle \end{aligned}\right\} \Rightarrow M = \sqrt{\tfrac{2}{9}}\, M_3 - \sqrt{\tfrac{2}{9}}\, M_1$$

(b) According to results received in part (a):

$$\sigma_a : \sigma_b : \sigma_c : \sigma_d : \sigma_e : \sigma_f : \sigma_g : \sigma_h : \sigma_i : \sigma_j =$$

$$= |M_3|^2 : \frac{|2M_3 - M_1|^2}{9} : \frac{|M_3 + 2M_1|^2}{9} : \frac{|M_3 + 2M_1|^2}{9} : \frac{|2M_3 + M_1|^2}{9} : |M_3|^2 : 2\left|\frac{M_3}{3} - \frac{M_1}{2}\right|^2 :$$

$$: 2\left|\frac{M_3}{3} - \frac{M_1}{2}\right|^2 : \frac{2|M_3 - M_1|^2}{9} : \frac{2|M_3 - M_1|^2}{9} .$$

(c) According to part (b): and $M_3 \gg M_1$

$$\sigma_a : \sigma_b : \sigma_c : \sigma_d : \sigma_e : \sigma_f : \sigma_g : \sigma_h : \sigma_i : \sigma_j = 9 : 4 : 1 : 1 : 4 : 9 : 2 : 2 : 2 : 2$$