Understanding Student Answers In Introductory Physics Courses

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1 Introduction

Many problems in introductory physics courses require the student to formulate a system of algebraic equations. The correct answer can be expressed in many ways, using different variable names, different numbers of variables, and different numbers of equations that may contain numeric substitutions for variables. It is essential that an Intelligent Tutoring System (ITS) in physics be able to understand the student's submission before it can provide useful feedback. Understanding the student's answer involves being able to determine the mapping from the equations, variables and constants to the principles, properties and objects of the problem.

Many different approaches have been used to tackle this problem. Most of them consist of some type of restriction or constraint on what the student can or needs to do. The ANDES tutoring system [8, 9, 10] requires the student to precisely define every variable before it can it be used in an equation, thereby specifying the mapping. Defining every variable explicitly is a very tedious step for the users of the system. Our earlier work [6, 2, 3, 5, 4], enabled relaxing this requirement, but it could only be applied to problems where numerical constants were not part of the problem. This is not true for most problems currently assigned for homework at the introductory physics level. Other researchers [7, 1] have used tutorial dialogs as an additional step to elicit the mappings from the student, with some degree of success.

This paper describes how our technique has been extended to reason about systems of equations that contain a mix of variables and numerical constants. This significantly extends the applicability of our approach, as most assigned problems provide numerical values for some of the physical properties, and students are expected to derive the numerical value of a specified property.

The technique has been evaluated on student submissions to three problems from the ANDES physics tutoring system [8]. The ANDES system has been in use for many years in the introductory physics course at the United States Naval Academy. The evaluation was carried out on student submissions from the fall semester of 2001. The evaluations show that the technique can effectively determine (1) the mapping of the student's variables to the objects and properties in the problem, (2) the correctness of each student equation, and (3) the completeness of the student's submission.

2 Problems and Issues

In this section we describe and discuss the problems and issues that arise when reasoning about systems of algebraic equations. We use two examples from an introductory physics course to illustrate the issues and concepts.

2.1 Two Example Problems From Introductory Physics

The following problems require the student to combine the application of the distance-time-speed kinematic relation to two separate time intervals. Students at the United States Naval Academy were presented with these problems, (Exkt3a and Exkt4a) when they used the ANDES system in the fall semester of 2001.

Exkt3a: Suppose you fire a bullet (speed 1600 m/s) in a shooting gallery and hear the gong on the target ring 0.731 seconds later.

Taking the speed of sound to be 330 m/s and assuming the bullet travels straight down-range at a constant speed, how far away is the target?

Exemplar	Student Submission
$d_b = d_s$	$v01 = s01/t01$
$d_b = v_b * t_{12}$	$s01 = v01 * t01$
$d_s = v_s * t_{23}$	$v01 = 1600 \text{ m/s}$
$t_{13} = t_{12} + t_{23}$	$v12 = 330 \text{ m/s}$
$v_s = 330 \text{ m/s}$	$t02 = .731 \text{ s}$
$t_{13} = 0.731 \text{ s}$	$v12 = s12/t12$
$v_b = 1600 \text{ m/s}$	$t01 = s01/v01$
	$v12 * t12 = v01 * t01$
	$t02 = t12 + t01$
	$t02 = v12{*}t12/v01{+}t12$

Exkt4a: A motorcyclist rides onto a road at 22.2 m/s at 12:00 noon and maintains that speed for the rest of her journey. At 13:00 a car traveling at 27.8 m/s turns onto the road at the same point as did the motorcycle.

How many seconds after the motorcycle started will it be passed by the car?

Exemplar Solution
$dslow = dfast$
$dslow = vslow * t0$
$dfast = vfast * t2$
$t0 = t1 + t2$
$t1 = 3600 \text{ s}$
$vslow = 22.2 \text{ m/s}$
$vfast = 27.8 \text{ m/s}$

Student Submission
$v_m = 22.2 \text{ m/s}$
$t_c = (v_m * 3600\text{s})/(5.6 \text{ m/s})$

The exemplar solutions and examples of student submissions are shown. The students were asked to specify the system of equations that would solve the problems, in addition to working out the answer. These student submissions are correct and complete, in the sense that every equation is correct and the solution can be determined from the equations given. But the student formulations differ from the exemplars, highlighting some of the many issues that have to be resolved by a physics ITS. These include

- *Dimensional Analysis*: Determining the dimensions of the variables, constants, and equations is an important initial step in constraining the mapping of the properties, objects and physics equations described by the student. Note that the problems do not specify the variables that are to be used, so the student is free to choose what variables she needs, and their names.

- *Mapping the Submission to the Exemplar*: A tutor cannot evaluate a student's submission without an exemplar set of equations and variables, and a mapping from the student variables into those of the exemplar. The exemplar can either be provided by the author of the problem (as we have done) or can be generated by the system (ANDES' approach). In either case, the mapping from the student's names to the exemplar's is difficult due to

  - *choice of variable names:* The mapping of the variables is complicated by the fact that the choice of variable names is left up to the student. There are also many different ways to specify the solution. In the Exkt3a problem, the student has used the same seven variables as the exemplar, though with different names, but in Exkt4a the student has used only two variables and two equations, but nontheless has a complete solution.

- *representation of constants:* Students frequently simplify terms by combining numerical values, for example multiplying or dividing by 2. The 5.6 m/s in the student's solution to Exkt4a appears nowhere in the problem statement, but is the relative velocity ($vfast - vslow$). The tutor needs to recognize this combination. Another complicating factor with constants is that they are only approximately represented on digital computers, so all the problems associated with finite precision arithmetic must be dealt with.
- *multiple equivalent systems of equations:* There are many different ways to specify a system of equations. In the examples, both the exemplar and the student submission are correct and complete. Each exemplar solution has four equations beyond the assignment statements (which simply set a variable to a given numerical value), while the student submissions have seven equations and one equation respectively, beyond assignment statements.
- *redundant equations:* Students frequently have redundant equations, i.e., they are reformulations of previous equations. For example, the second equation in Exkt3a is a rewrite of the first equation. An ITS must be able to detect redundant equations in the process of determining if the student submission is complete, i.e., contains all the necessary equations.

3 Approach/Algorithm

Our approach is based on mapping the equations and variables in the student's submission to the equations and variables in an exemplar solution. This approach was described in [5] and evaluated on a set of unrestricted handwritten answers from students in an introductory physics course. Students were not asked to define the variables they used in the equations. The questions did not use any numerical values and thus the student's answers did not contain any numbers with dimensions. The evaluation showed that our technique was generally successful in mapping such submissions.

However, most problems in introductory physics courses use dimensioned constants and also ask the student to solve for the value of a variable that represents a physical property. We have extended our approach to accept answers to problems that use dimensioned constants. The approach uses the following sequence of steps:

1. *analysis of the exemplar solution:* The analysis[4] determines the dimensions of each variable, constant and equation from the exemplar solution. This analysis relies strongly on (1) identification heuristics based on the common use of variable names to represent corresponding physical properties (e.g., variables starting with v typically represent velocity and variables starting with T typically represent either time or force), and (2) a constraint propagation mechanism to propagate information about the constraints on the viable dimensions for each variable, constant and equation.

2. *analysis of the student submission:* The analysis determines the dimensions of each variable, constant and equation from the student's submission.

3. *identify numerical values:* The numbers (and corresponding variables) from the student submission are mapped to the constants (or combination of constants) from the exemplar solution. The constants and variables in the student submission are replaced with an equivalent combination of variables from the exemplar. For example, in the second example, Exkt4a, shown in Section 2.1, two velocities (22.2 m/s and 27.8 m/s) are used in the exemplar, but student submissions frequently use only their difference (5.6 m/s).

   The algorithm matches and identifies numerical values using the following steps:

   (a) *build a list of numerical values from the exemplar*: The information in this list includes (a) the value, (b) the dimensions and (c) the variable that the value is assigned to.

   (b) *find direct match of student's numerical values*: Each numerical value in the student's equations is compared against the numerical value in the previously built list. A match occurs if the two values are within 1% and the dimensions are the same. In addition, the variable that the student sets to that value (if present) is then replaced throughout the system of equations with the corresponding variable from the exemplar. This helps to reduce the complexity when trying to map the student's variables and equations to the variables and equations of the exemplar.

   (c) *match composite values*: The remaining unidentified numerical values are then compared against values formed as algebraic compositions of the numeric values given in the exemplar and simple integers.

   If a match is found (within 1%), the numeric value is replaced by the equivalent combination of variables. Three types of algebraic composition are analyzed:
   1) inverses of the given values
   2) combinations under addition and subtraction of values which have the same dimension
   3) combinations of two values by multiplication or division.

   Having replaced all the constants with the variables they represent, we are left with a purely algebraic system of equations with only dimensionless constants.

4. *map variables and equations from the student submission:* The resulting variables and equations (see previous step) from the student submission are mapped to corresponding variables and equations from the exemplar solution.

   This step requires the algorithm to first transform the single system of equations in the exemplar into equivalent systems of equations with fewer variables, thus generating all equivalent correct solutions. When we have a version of the exemplar that has the same number of equations and variables of each dimensionality as the student submission, we then try and find a mapping of the variables and equations from one to the other. The mapping algorithm used (from [5]) is:

   > A matrix of dimension signatures is constructed for both the solution set and the student set of equations. The goal is to find one or more correct mappings between the variables and equations of the two sets. A mapping between the two matrices is correct if the matrices are identical, i.e., every entry in one matrix is identical to the corresponding entry in the other matrix and the *given*[1] variables are in the same columns in both matrices. When

[1] The *given* variables are those explicitly named in the problem presentation.

this happens, we have a *dimension map* between the student solution and the exemplar. Possible mappings are generated by permuting the rows and columns of the solution matrix.

If this step fails, i.e. the student submission is incorrect, the algorithm incrementally removes equations systematically from both the student submission and the exemplar and then matches the remaining equations. The purpose of this step is to find the maximum number of student equations that match to equations in the exemplar. The equations in the exemplar solution that do not map to student equations point to either (1) equations that the student is unable to generate or (2) equations that the student has generated incorrectly. This information is used to guide the generation of useful feedback to the student.

4 Evaluation

We evaluated our technique on data from three problems from the ANDES system. The problems were presented to students at the United States Naval Academy in the fall of 2001. The three problems all involve the use of physics principles from classical mechanics with the limitation that the solutions do not involve any trigonometric functions.

1. *Problem 1 - Exkt3a*: This problem was described previously (Section 2.1) and involves reasoning about speed, time and distance.
2. *Problem 2 - Exkt4a*: This problem was also described previously in Section 2.1 and also involves reasoning about speed, time and distance.
3. *Problem 3 - Exe5a*: This problem involves reasoning about conservation of energy using the potential and kinetic energy of a single object.

> *Exe5a*: A frictionless roller-coaster car tops the first hill whose height is 30.0 m above the base of the roller coaster with a speed of 3.00 m/s
>
> What is the magnitude of the instantaneous velocity at the apex of the next hill which has a height of 10.0m?
>
> Choose the zero of gravitational potential energy at the base of the roller coaster.

4.1 Experimental Results

We evaluated our technique on the three problems from the fall semester of 2001 of an introductory physics course at the United States Naval Academy. The data we extracted from the ANDES logs contained only correct equations. While the individual equations are correct, the student submission may be incomplete or contain redundant equations. Because ANDES immediately informs students if their equation is incorrect, and encourages the student to immediately fix it, the available logs did not provide full submissions including wrong equations which were appropriate to include in our test. Our earlier experiments [5] evaluated the ability of the algorithm to detect and analyze incorrect equations to generate useful feedback, but these evaluations were restricted to pure algebraic equations, i.e., they did not contain dimensioned numerical values. Thus, the current evaluation is incomplete in that we cannot evaluate how well the algorithm works on incorrect numerical values and more importantly,

	Exkt3a	%	Exkt4a	%	Exe5a	%
Successful analysis	210	**89.8%**	250	**96.2%**	135	**93.2%**
a) Correct Solution	83	35%	118	45.4%	118	81.4%
b) Partial Solution	110	47%	99	38.1%	11	7.6%
c) No nontrivial equations	17	7.3%	33	12.7%	6	4.1%
Unsuccessful analysis	24	**10.2%**	10	**3.8%**	10	**6.8%**
d) Non-derived equations	6	2.5%	0	0%	4	2.7%
e) Unknown constants	18	7.7%	10	3.8%	6	4.1%
Total	234	100%	260	100%	145	100%

Table 1: Results from ANDES Fall 2001 data

whether it can determine some plausible causes (misconceptions about physics concepts) for the errors.

The results of the evaluation are shown in Table 1. Each column shows the number of submissions for each problem in each category and next to it is the percentage. The last line in the table shows the total number of submissions for each problem. The student submissions are broken down into two major categories: (1) those that our technique could fully analyze and (2) those that it could not. The submissions that were successfully analyzed were further broken down into three subcategories. Category (a) are the submissions that were determined to be correct and complete solutions to the problem. Category (b) are the submissions that contained some number of correct equations but did not contain sufficient equations to solve the problem. Category (c) are the submissions where the student did not enter any equations beyond the the assignment of explicitly given values to variables.

The submissions that could not be fully analyzed fall into two categories. Category (d) are submissions that contain equations that could not be resolved because of limitations with the implemented technique, e.g., common factors have been omitted (Section 4.2). Category (e) are submissions that used numerical values that were **not** recognized by the algorithm for a variety of reasons. These reasons are discussed in the next section.

The results from Table 1 show that the technique was successful in parsing and analyzing over 90% of the student submissions (89.8% for Exkt3a, 96.2% for Exkt4a and 93.2% for Exe5a).

4.2 Discussion

There are several limitations to our approach. These limitations result in (1) the submissions that could not be analyzed (see Table 1) and (2) problems that we could not evaluate. The main limitations are:

- *trigonometric functions:* Trigonometric functions add a great deal of complexity. The system must understand and manage general trigonometric identities (students will use every possible equivalent form), and in addition, it must recognize common substitutions of *equivalent* values. For example, if the problem includes a right triangle with a $37°$ angle many students will recognize this as a 3-4-5 triangle and take $\sin 37 = 3/5$. There are many other similar situations.

- *constants with a value of zero:* This occurs frequently in conservation of momentum problems where one of the bodies is initially at rest (velocity = 0). The common simplification is to omit terms where a zero velocity is involved but this is not always done. The solution is to have the system generate a second exemplar solution where the terms that have a zero are omitted. The system will then match a student submission against both exemplars.

- *unrecognized constants:* Students will use combined constants in their equations, also possibly combining the given constants with factors like 2 or 1/2 which occur in the instantiation of physics principles. Our heuristic methods are able to detect most such combinations.

- *non-derivable equations:* Students also will use algebraic simplifications that do not have a corresponding physics model. The two equations listed below are each correct solutions to problem Exe5a (see previous section).

$$m * g * h_1 + 0.5 * m * v_1^2 = m * g * h_2 + 0.5 * m * v_2^2 \tag{1}$$
$$g * h_1 + 0.5 * v_1^2 = g * h_2 + 0.5 * v_2^2 \tag{2}$$

  Equation 1 can be derived using algebraic substitution on the exemplar but equation 2 (student submission) cannot be derived. In this instance, the student has removed the common factor m from equation 1 resulting in equation 2. Equation 1 deals with conservation of energy ($\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2$) but equation 2 has the dimensions of $\mathrm{m}^2/\mathrm{s}^2$ which has no physical correspondence.

  The solution is to generate a second exemplar where the common factors are removed. The system will then match student submissions against both exemplars.

It is possible to extend our algorithm to address the previous limitations. A generic method that *guarantees* success would increase the computational cost of the analysis and prevent a system from providing feedback in real-time. However, we believe that effective heuristics, possibly problem specific heuristics, can address the most frequently occurring problems and perform well enough to provide real-time feedback.

5 Conclusion

We have described how a student's algebraic equations in physics can be analyzed so that their errors can be detected and useful feedback provided. Unlike previous approaches, our approach does not require the student to define the variables explicitly (a time consuming step) and can analyze problems with embedded dimensional numeric constants commonly used in introductory physics courses. The technique has been evaluated on three problems from the ANDES system. The evaluation shows that the technique is successful over 90% of the time.

6 Acknowledgments

We are grateful to Kurt VanLehn and the Andes group for making the ANDES logs available to us, and to Anders Weinstein for answering questions about the ANDES implementation. Jim Appenzeller and Dave Santin provided invaluable help in the implementation of large parts of the system.

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