What Is The Student Referring To? Mapping Properties and Concepts in Students' Systems of Physics Equations

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Abstract.

An ITS dealing with students' algebraic solutions to Physics problems needs to map the student variables and equations onto the physical properties and constraints involved in a known correct solution. Only then can it determine the correctness and relevance of the student's answer. In earlier papers we described methods of determining the dimensions (the physical units) of student variables. This paper describes the second phase of this mapping, determining which specific physical quantity each variable refers to, and which part of the set of constraints imposed by physics principles each student equation incorporates. We show that knowledge of the dimensions of the variables can be used to greatly reduce the number of possible mappings.

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1. Introduction

Many problems in introductory physics require the student to provide a set of equations as an answer. Such an answer is composed of several components including equations, variables, and mathematical operators. An Intelligent Tutoring System (ITS) for physics must *understand* the student's submission in order to generate useful feedback. In particular, it must determine the physics principle used in each equation and to which properties and objects each variable refers. This is difficult when (1) there are many possible ways to specify a correct answer, (2) there are many reasonable names for variables that represent properties (*e.g.*, the mass of object 1 could be $m1$ or m_1 or m), or (3) the student submits an incorrect answer.

This paper describes a technique that reasons about all components of a student's submission to determine a correct interpretation. The approach taken is to compare the student's submission to a recorded correct solution for the problem (*i.e.*, the *exemplar*). If the student submits a correct solution and that solution is, equation by equation and variable by variable, a rephrasing of the the exemplar, the solution can be validated by identifying the mapping between the student's and exemplar's variables and equations. The number of possible mappings can be very large; however, the complexity of the search can be effectively managed when the dimensions of the student variables are known or can be determined.

Experience has shown that even correct answers seldom have a simple correspondence to an exemplar. Submissions that look very similar to an exemplar can be symptomatic of a misunderstanding of physics while those that look very different can be seen as correct once the concepts represented by the variables and equations are understood.

Consider a problem based on Atwood's machine, a frictionless pulley with two masses, m_1 and m_2 hanging at either end. A simplified[1] exemplar solution consists of

$$T_1 - m_1 * g = m_1 * a_1 \tag{1}$$

$$T_2 - m_2 * g = m_2 * a_2 \tag{2}$$

$$T_1 = T_2 \tag{3}$$

$$a_1 = -a_2 \tag{4}$$

Table 1 shows three possible submissions to the problem. The exemplar contains four equations but none of the three submissions contains more than three equations. Submission A is an incorrect solution that can result from a misunderstanding of how the direction of a vector effects a result. Submission B is a correct solution and can be derived by algebraic simplification of the exemplar. Submission C introduces a new variable that is not found in the exemplar. It cannot be derived by an algebraic simplification of the exemplar, but it is correct if M is understood to represent $m_1 + m_2$.

Previous approaches have either (1) severely constrained the student input to use pre-specified variable names[5], or (2) used *strong scaffolding* to force the student to define the referents of her variables[7], or (3) used heuristic techniques to map the variables and equations[4]. Our algorithm considers all possible mappings of the student's variables and equations onto the exemplar, and computes the *distance* between the image and pos-

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[1]The full exemplar solution used in the experiment of section 4 contains eight equations, with the additional variables W_1, W_2, $Fnet_1$ and $Fnet_2$.

Submission A	Submission B	Submission C
$T - m_1 * g = m_1 * a_1$ $T - m_2 * g = m_2 * a_2$ $a_1 = a_2$	$T - m_1 * g = m_1 * a$ $T - m_2 * g = -m_2 * a$	$a = (m_1 - m_2) * g/M$

Table 1. Several Possible Submissions for Atwood's Machine

sible algebraic reductions of the exemplar set. If that fails to give a full match, equations are dropped from the student and exemplar sets to find the best mapping. If there is more than one best mapping, heuristics are used to select a mapping. The selected mapping is used to evaluate the submission for correctness and to identify possible errors.

2. Algebraic Physics Problems

An ITS for physics must first determine (a) what physics property (*e.g.* force, momentum) each variable represents and (b) to which object or system the property applies and at what time. Only then can the ITS determine if (c) each equation is relevant and correct and finally (d) if the set of equations is correct and complete. Some ITS's like ANDES [8,7] solve problems (a) and (b) by strong scaffolding that requires the student to define each variable, *i.e.* specify its dimensions and the object it applies to, before it is used. The system then uses its knowledge of the variables to determine the correctness of the equations using a technique called "color-by-numbers" [7,6]. In earlier papers [1,2,3] we described an alternative technique that determined the dimensions of students' variables from the context of the equations, thus solving issue (a). This paper describes our current work on solving issues (b), (c) and (d). We illustrate the problems involved with an example problem based on Atwood's machine, as shown in Figure 1a.
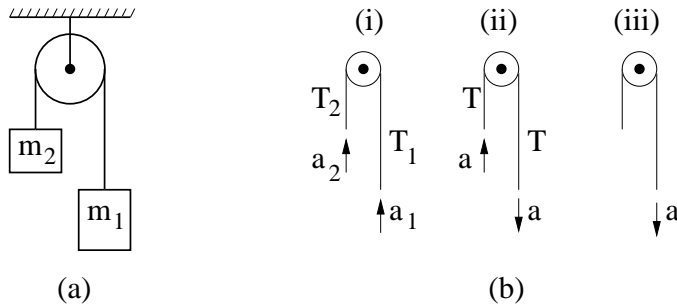

Figure 1. Atwoods Machine

A common problem based on Atwood's machine asks the student for the equation(s) that would determine the acceleration of the mass m_1, assuming that m_1 and m_2 are not equal. Equations 1 through 4 represent a correct solution using variable set (i) in Figure 1b.

In an alternative formulation, the student chose to use a single variable a to represent acceleration and a single T for the tension. She implicitly used the principle that equates

T_1 and T_2, and the constraint $a_1 = -a_2$, which comes from the fixed length of the cord. Variable set (ii) in Figure 1b identifies the variables used with such an approach. The resulting equations are "Submission B" in table 1.

In comparing the student's equations with the exemplar solution, an ITS must determine the mapping of the variables and equations from one set to the other. This process is complicated by several issues:

1. *variable renaming:* The student and the instructor may use different variable names to represent the same quantities. There is no restriction on the names of variables or choice of subscripts even though there are many standard variable names. There are also many commonly used variations, e.g. F, $Fnet$, F_1 can represent the same force.
2. *simple aliasing of one variable:* Frequently, variables that have the same magnitude and dimensions are aliased for one another. For example, the variables T_1 and T_2 in equations 1, 2 and 3 are equal to one another. In submission B of table 1, there is only a single variable T that is used to represent both, i.e., T is an alias for both T_1 and T_2. When variables are aliased, the number of equations in the set is reduced.
3. *elimination by solution for variables:* There are many ways to specify the algebraic solution to a problem. These may involve using a greater or lesser number of variables and thereby a greater or lesser number of equations. For example, one very different but correct solution to the example problem is:

$$m_1 * g - m_1 * a = m_2 * g + m_2 * a$$

   In this case, there is no variable representing the tension of the rope (commonly T, T_1 or T_2). Instead that variable has been solved for in one equation, which is eliminated from the set, and then substituted for in the other equations.

These issues result in there being many possible mappings between the variables and equations of a student's submission and that of the exemplar solution. Systems like ANDES [8,7] require that the student specify the mapping of variables. A mapping of equations (if it exists) can then be more easily derived. If the student input is not constrained in this way, the ITS must deal with the computational complexity issues. If each equation is evaluated singly, then each evaluation results in many possible interpretations and requires the use of strong heuristics to select a correct mapping [4]. Our algorithm considers all the variable and equation mappings simultaneously. The combination of all constraints greatly reduces the number of possible mappings that must be considered.

3. The Mapping Algorithm

The algorithm identifies properties and concepts by finding mappings of the variables and equations from a student set of equations to the variables and equations in an exemplar solution. The variables and equations in the exemplar are annotated with their dimensions and the associated physical principle [3].

The mappings of variables and equations are interdependent and the algorithm simultaneously finds a mapping for both variables and equations. This section describes

how the dimensions of the variables are used to find the variable and equation mappings. Sections 3.1.2 and 3.3 show how the mappings can then be used to determine the algebraic differences between the student's equations and the exemplar.

3.1. Matching Dimensions

The dimensions of the variables are used to infer the dimensions of the equations. Each equation has a signature consisting of the dimensions of the equation and a vector of 1's and 0's, where a 1 indicates this equation contains the corresponding variable. Similarly, the signature of a variable consists of the dimensions of the variable and a vector of 1's and 0's, where a 1 indicates this variable is contained in the corresponding equation. The signatures are combined together to form a matrix where each row is the signature of an equation and each column is the signature of a variable (Table 2).

	T1	T2	m1	m2	a1	a2	g	dimension
Eqn 1	1	0	1	0	1	0	1	$kg \cdot m/s^2$
Eqn 2	0	1	0	1	0	1	1	$kg \cdot m/s^2$
Eqn 3	1	1	0	0	0	0	0	$kg \cdot m/s^2$
Eqn 4	0	0	0	0	1	1	0	m/s^2
dimension:	$\dfrac{kg \cdot m}{s^2}$	$\dfrac{kg \cdot m}{s^2}$	kg	kg	$\dfrac{m}{s^2}$	$\dfrac{m}{s^2}$	$\dfrac{m}{s^2}$	

Table 2. Matrix of signatures for Equations 1 through 4

3.1.1. Comparison of Matrices

In this section we assume that the exemplar and the student set of equations have the same number of equations and variables of each dimensionality. A matrix of dimension signatures is constructed for both the solution set and the student set of equations. The goal is to find one or more correct mappings between the variables and equations of the two sets. A mapping between the two matrices is correct if the matrices are identical, *i.e.* every entry in one matrix is identical to the corresponding entry in the other matrix and the *given*[2] variables are in the same columns in both matrices. When this happens, we have a *dimension map* between the student solution and the exemplar. Possible mappings are generated by permuting the rows and columns of the solution matrix subject to the following constraints:

- Rows (equation signatures) can be interchanged only if the equations have the same dimensions.
- Columns (variable signatures) can be interchanged only if the variables have the same dimensions.

In Table 2, if dimensions are ignored there are $4! \times 7! \, (= 120,960)$ possible permutations. If we restrict row and column interchanges to those with the same dimensions then rows 1,2 and 3 can be permuted, columns 1 and 2 can be interchanged, columns 3 and 4 can be interchanged and columns 5, 6 and 7 can be permuted. The set of four equations (Equations 1 through 4) can yield 144 different permutations (mappings of vari-

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[2]The *given* variables are those explicitly named in the problem presentation.

ables and equations) that are dimensionally equivalent. We can further restrict the inter-changes such that rows (equations) can only be interchanged if they use the same number of variables of each type. Applying this restriction to both row and column interchanges as well as constraining the given variables to be in the same columns in the exemplar and student matrices further reduces the number of permutations to 8. This technique when applied to the full exemplar solution for Atwood's machine (8 equations) reduces the number of permutations by a factor of 100 million from $8! \times 11! = 1.61$ trillion to $2! \times 4! \times 2! \times 2! = 9216$.

3.1.2. Evaluation of Equations for Correctness

The dimension information significantly reduces the search space but it is not sufficient to determine if the equations are correct. One of the many techniques for determining correctness, developed by Shapiro [6,7] and used in the ANDES system, is to instantiate the variables with random consistent values and then evaluate the equations to see if the values hold. This method, while effective for correct equations, does not help in identifying the causes of errors in equations. Our technique instead compares the mapped student equations with the corresponding equation from the solution set, term by term, to find the algebraic differences between the equations. This requires that the equations in both the solution set and the student set be represented in a canonical form as a sum of products. The algebraic differences (errors) that can be detected include (1) missing terms, (2) extra terms, (3) incorrect coefficients and (4) incorrect signs (a '+' instead of a '−' and vice versa). For this technique to be generally applicable and successful, it must also take into account differences that are not errors, such as various orderings of terms or factors, and multiplication of the entire equation by a constant. The algebraic differences are then used to identify the physics principles that have been incorrectly applied.

3.2. Dealing with Equation Sets with a Different Number of Equations/Variables

It is often the case that students will generate answers that contain a different number of variables through the use of algebraic transformations. The matching algorithm uses the exemplar solution to construct a lattice of equivalent sets of equations that contain a smaller number of equations and variables. Construction of the lattice proceeds as follows from the exemplar equations:

1. Initialize the lattice with the exemplar and mark it on the frontier.
2. The equations in each node on the frontier of the lattice are analyzed for variables that can be solved for in terms of other variables. Variables whose values are specified (givens) or that the student is supposed to find (the goal) are excluded.
3. Substitute for the variable in each of the other equations in the node. This results in a new set of equations with one fewer equation and forms a new node in the lattice.
4. This process (steps 2 and 3) is repeated until the nodes on the frontier all contain only one equation for the goal variable.

The student's set of equations is then compared (Section 3.1.1) against the equations from nodes in the solution lattice that have the same number of equations and variables of each dimensionality. All valid mappings are collected into a list of possible mappings which are then used to evaluate the student's set for correctness (Section 3.1.2). If there

is a mapping that results in the student's equations being evaluated as correct, then the student's equations are marked correct.

3.2.1. Application of Substitutions

Substitutions are applied only to the solution set of equations and not the student's set. This allows the system to refer to the student's original equations when generating feedback. In addition, this restriction greatly reduces the number of possible mappings. An exemplar set of 8 equations, if we ignore repetitions, results in a lattice that contains 2^8 nodes.

This approach works only if the exemplar solution encompasses all the correct variations that the student might use. If the student uses a variable that is not in the solution set (*e.g.* submission C in Section 1), the algorithm will not be able to (a) find a map or reference for the variable (b) evaluate the equation for correctness.

3.3. Matching Incorrect or Incomplete Equation Sets

The algorithm has been extended to determine the mappings even when there are equations that are missing, extra, incorrect or irrelevant. This phase of the algorithm is executed when a complete dimension match of the variables and equations cannot be found.

Equations are systematically removed one at a time from the exemplar and/or the student set of equations. After removal of the non-matching equations, the matching algorithm (Section 3.1.1) can be used to match the remaining equations and variables. The variable maps that are found from the match can then be used to try to derive the complete variable maps.

The algorithm starts by taking each node in the lattice of correct solutions (Section 3.2). and making it the top of a new lattice where all the other nodes contain incomplete sets of equations with one or more missing equations. This results in many lattices with incomplete sets of equations except for the top of each lattice. A similar lattice of incomplete sets of equations is constructed for the student's set of equations. Starting from the top of the student lattice, the algorithm compares each node with the equivalent nodes (ones with the same number of equations and variables of each dimensionality) from the lattice of lattices created from the exemplar. The comparison stops after trying to match all nodes at a level in the student lattice if any dimension match is found (Section 3.1.1). These matches are then applied to the student's variables and equations to give a set that is evaluated for correctness (Section 3.1.2).

4. Experiments

We collected answers to four pulley problems from 88 students in an introductory physics course. One of the four problems was the Atwoods problem and our initial evaluation focused on the answers to that problem. The students were not restricted in any way except that they were asked to refrain from making algebraic simplifications to their answers. The student answers were evaluated against the exemplar with 8 equations and the results are described below.

- Five equation sets were dimensionally inconsistent.

- Three equation sets were dimensionally ambiguous. Dimensional ambiguities frequently arise when the students only enter a single equation. The single equation does not provide sufficient context for the system to uniquely determine the dimensions of one or variables.
- 47 equation sets matched using substitutions and the matrix of dimensions. These were further broken down into:

  * 31 equation sets that matched exactly when compared term by term with a node in the exemplar solution set lattice.
  * 16 equations sets that had algebraic differences consisting of either (1) an incorrect sign, (2) an extra term or (3) a missing term.

- 22 equation sets dimension-matched partially, *i.e.* only after elimination of one or more equations from either the student or solution set of equations. Six of the 22 equation sets had extra equations. The algorithm was able to identify the extraneous equations as well as determining that the remaining equations were both correct and complete.
- 11 sets of equations had no non-empty subset that matched (dimensionally) any equations in the solution set.

For comparison, we used the ANDES algebra subsystem [6] to evaluate the same set of equations. In this case, we had to define the variables explicitly before evaluating each set of equations. The ANDES system found the same results as our algorithm except for one instance where the student used a correct but non-standard formulation (submission (C) in Section 1). In this one case, applying substitution (Section 3.2) on the student set of equations would have resulted in the algorithm discovering that the answer was correct. ANDES would not have permitted the student to define or use the variable M.

4.1. Discussion

The results show that the algorithm performed as well as the ANDES system on the equation sets that both could solve. This indicates that the combination of our earlier algorithm for determining the dimensions of variables and this algorithm for matching equations and variables may be sufficient to relax the scaffolding, not requiring the student to explicitly define variables before using them. In addition, the algebraic differences detected will facilitate generation of specific and useful feedback to the student.

The technique is most successful when the student uses a larger number of equations, *i.e.* minimizes the use of algebraic simplifications. The additional equations provide a context that enables the technique to efficiently find the correct mapping of variables and equations in most instances. When a correct mapping can be found, the algorithm finds either one or two mappings and if there are two or more mappings, heuristics are used to select one. The algorithm has been shown to be effective on the example problem as it reduces the possible mappings to just one or two correct mappings.

The algorithm relies on the student using variables that can be mapped onto variables from the exemplar solution. This does not always happen, as in the case of submission (C) in Section 1. In those cases, we can apply the substitution algorithm to the student equations as well. This is applied as a last resort because (a) the number of possible matches grows very quickly and (b) it is difficult to generate reasonable feedback.

5. Conclusion

We have described a technique that determines the objects (and systems of objects) and properties that variables in algebraic equations refer to. The algorithm efficiently uses the dimensions of the variables to eliminate most of the possible mappings and find either one or two correct mappings which can then be further refined with heuristics. The technique is effective even if the student's answer uses a different number of variables and equations than the solution set. The mapping of variables and equations has been used to determine the algebraic differences between the student's answer and the solution set. This can lead to more effective feedback when the student's answer is incorrect. The technique has been evaluated on a small set of answers to one specific question and compares well with the results of a well-known system (ANDES) that uses much tighter scaffolding.

Acknowledgments

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