Review

Last time we saw that if a physical system (*e.g.* Hamiltonian) has a symmetry, the solutions can be transformed by that symmetry to form possibly new solutions. In particular, for a quantum-mechanical system the space of solutions with a given energy is a linear subspace which is a **representation** of the symmetry group.

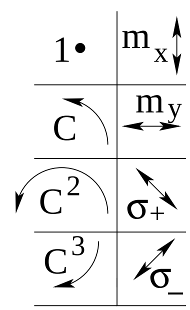
We defined the mathematical notion of a **group** as a set G with a binary operator $\odot : G \times G \to G$ which is closed, associative, having an identity and each element has an inverse. Groups can be finite or infinite, and the latter can be discrete or continuous. The "multiplication" law $\odot$ can be commutative or not. If a subset of G satisfies the definition of a group by itself, it is a **subgroup**. Every group has itself as a subgroup, and also the subgroup consisting only of the identity. Other subgroups are called **proper subgroups**.

A finite group's multiplication can be defined by a multiplication table. Of necessity, each row and each column in that table contains each element of the group exactly once. This leads to the **rearrangement theorem**

$$\sum_{A\in G} f(A) = \sum_{A\in G} f(AB), \text{for any } B \in G.$$

We considered the symmetry group of the square, C_{4v}, with 8 elements, and worked out its multiplication table. Here is the multiplication table we worked out:

$A \backslash B$	$\mathbb{1}$	C	C^2	C^3	m_x	σ_+	m_y	σ_-
$\mathbb{1}$	$\mathbb{1}$	C	C^2	C^3	m_x	σ_+	m_y	σ_-
C	C	C^2	C^3	$\mathbb{1}$	σ_+	m_y	σ_-	m_x
C^2	C^2	C^3	$\mathbb{1}$	C	m_y	σ_-	m_x	σ_+
C^3	C^3	$\mathbb{1}$	C	C^2	σ_-	m_x	σ_+	m_y
m_x	m_x	σ_-	m_y	σ_+	$\mathbb{1}$	C^3	C^2	C
σ_+	σ_+	m_x	σ_-	m_y	C	$\mathbb{1}$	C^3	C^2
m_y	m_y	σ_+	m_x	σ_-	C^2	C	$\mathbb{1}$	C^3
σ_-	σ_-	m_y	σ_+	m_x	C^3	C^2	C	$\mathbb{1}$



Elements of C_{4v}

Today

Today we will discuss relations between the elements of a group and between groups, and how to make groups from others. We saw last time that a few elements can **generate** the rest of the group, so that C generates the four element group C_4, which is therefore a proper subgroup of C_{4v} and throwing in any one of the reflections is enough to generate the rest of C_{4v}. We will discuss mappings between groups which preserve the group multiplication rules, which are called **homomorphisms** or, if 1-1, **isomorphisms**. Two groups which are isomorphic are generally considered to be the same group, even if their physical contexts differ.

We will be concerned with how the group elements can act on the vector spaces of states, which is to say representations of the group, and to help us find those possibilities we classify the elements into **conjugacy classes**, elements within each class are **conjugate** to each other, conjugacy being a form of **equivalence**. Another way of grouping elements occurs if there is a proper subgroup, so we can group together elements which are related by the subgroup — these are called **cosets**. If it doesn't matter whether we relate the elements by subgroup elements from the left or from the right, the subgroup is called a **normal subgroup**, and then the cosets form another group called the **quotient group**. We can also take the **direct product** of two groups to form a larger group, and sometimes, but not always, the direct product of the normal subgroup with the quotient group is isomorphic to the original group.

Every homomorphism from $G \to K$ maps the identity of G into the identity of K, but there may be other elements of G which are also mapped into $\mathbb{1}_K$. The set of all such elements is called the **kernel** of the homomorphism and it is a normal subgroup of G.

We will discuss some important groups and some simple examples.

The **permutation group** on n objects, S_n, plays a very big role in constructing representations. Any permutation can be built up from **transpositions**, permuting two elements at a time, and it turns out that, although there are many ways to get a given permutation, they all either use an even number or use an odd number of transpositions, so that we may define a map $S_n \to \mathbb{Z}_2$ mapping those using an odd number into -1 and those using an even number into 1. The kernel of that map, that is those built from a even number, is clearly closed and therefore a group itself, and is called the **alternating group** A_n.

Just for examples, we will find all the groups of order 6 or less.