Physics 618 Twenty-third Lecture April 14, 2017

Last time, after finishing up our discussion of gauge groups on a lattice and motivating the group-invariant Hurwitz measure, we briefly described lattice translational symmetry, phonons and Bloch functions, and then began the discussion of spontaneous symmetry breaking. For a single particle Hamiltonian with a symmetry, we know that the eigenstates will be described by irreducible representations of the symmetry group, and even if the classical lowest energy states are not symmetric, the ground state of the quantum system will be. In particular, if we consider the wine-bottle potential

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4,$$

which has minimum energy states at $\phi = \pm|\mu|/\sqrt{\lambda}$, the lowest energy quantum states are the symmetric or antisymmetric superposition of wave functions centered around those two points, but with a slight energy difference depending on the overlap of the two wave functions.

If, however, we consider a lattice of such coordinates $\phi_{\vec{n}}$ with nearest neighbor couplings which discourage differences (as the discretization of $(\vec{\nabla})^2$ would give), the classical ground states would need all sites to have ϕ in the same direction, and the overlap of the state with all N sites up with that with all states down would be proportional to the small single-site overlap raised to the N'th power, and for an infinite lattice would be zero. That means that if initially the state of the system is finitely perturbed from the all-up state, there is zero overlap with the ones connected to the all-down state, and one might as well disregard all states from the Hilbert space, and the symmetry of the Hamiltonian is not reflected in the states we need to consider.

The model we just discussed with a single real field ϕ is essentially the Ising model, and its behavior in two dimensions, solved by Onsager, is a great triumph of mathematical physics. But I want to discuss a different consideration, considering a field ϕ which is not a single real variable but rather transforms non-trivially under a symmetry group, for example $SO(N)$, with the ϕ at each site transforming as under $e^{i\omega^b(\vec{x})L_b}$. The pure kinetic energy term $\frac{1}{2}m\sum_{\vec{x}}(\dot{\phi}(\vec{x}))^2$ and the potential term $\sum_{\vec{x}}V(\phi^2(\vec{x}))$ are invariant under both global and local rotations of ϕ, but the nearest neighbor coupling (or the gradient term in the continuum limit) will pick up an energy proportional to $(\vec{\nabla}\omega)^2$, and local symmetry is violated. This will lead us to Goldstone bosons, which in this case are spin waves. We will consider more generally a Hamiltonian which is invariant under a Lie group with Lie algebra $\mathfrak{G}$, but with a lowest energy state which is invariant only under a subgroup K with algebra $\mathfrak{K}$. We will find there is a Goldstone boson, that is a massless particle, for each direction in the coset space $\mathfrak{G}/\mathfrak{K}$

The same thing applies to a field theory. We first consider a simple global SO(N) scalar theory with the wrong sign for the quadratic (mass) term, and we see how all the component fields but one become massless. This gives a model which is sometimes used to describe pions.

Then we look into the magic that occurs when the symmetry group is a gauge group, so we have massless vector particles in our Lagrangian. But if there is spontaneous symmetry breaking in the matter fields, the massless Goldstone bosons can get eaten by the corresponding massless gauge particles, making them fat (massive). This is the Higgs mechanism, responsible for the large masses of the $W^{\pm}$ and Z^0 weak interaction mesons of the Salam-Weinberg model of electroweak interactions.