Last time, having decided that a latticized version of the field would help us understand the form of the pure gauge field term in the lagrangian, we found it would be in terms of the group elements coming from parallel transporting around placquettes. With U_P the adjoint representation of that group element, we found that the suitable term in the action involves $\mathrm{Tr}(U_P + U_P^\dagger - 2 \cdot \mathbb{1})$, summed over all placquettes in all directions. To determine the suitable Hamiltonian formulation, we first took the continuum limit in the time direction. Note that for any configuration of group elements on all the links of the lattice, we can do gauge transformations which leave all spacelike links at time $t = 0$ unchanged but convert each time link to the identity, to find a gauge-equivalent configuration of links with all time links set to $\mathbb{1}$. This is known as the temporal gauge. Then the placquette in the (0,j) plane has

$$U_{P_{0j}} = \underbrace{U^\dagger_{n^\nu,0}}_{\mathbf{1}} U^\dagger_{n^\nu+a^t,j} \underbrace{U_{n^\nu+a^j,0}}_{\mathbf{1}} U_{n,j}.$$

As the time lattice spacing goes to zero, we may Taylor expand $U^\dagger_{n^\nu+a^t,j}$ to second order in the lattice spacing a^t, and similarly for $U_{n^\nu+a^t,j}$. When adding, $U_P + U_P^\dagger - 2$, the first order terms are $a^t \frac{\partial}{\partial t}(U^{-1}U) = 0$, and the second order piece can be reexpressed as $\mathrm{Tr}(U_j^{-1}\dot{U}_j U_j^{-1}\dot{U}_j)$. The space-space placquettes will give a potential term $V(U_{P_{jk}})$ without time derivatives. Thus

$$L = \sum_\ell \frac{-a}{2g^2} Tr\left(U_\ell^{-1}\dot{U}_\ell U_\ell^{-1}\dot{U}_\ell\right) - V(\{U\}).$$

We will concentrate on the first term, trying to understand the kinetic terms that do involve time derivatives, and hence the canonical momenta. The canonical coordinates are matrices on each link, $U_{\ell\, ab}$, where ℓ specifies the link, both position and spacelike direction, and a, b are indices of Lie algebra generators.

Today

We will begin today with a naïve definition of the canonical momentum

$$(P_\ell)_{cb} := \frac{\delta L}{\delta\left(\dot{U}_\ell\right)_{bc}} = \frac{-a}{g^2}\left(U_\ell^{-1}\dot{U}_\ell U_\ell^{-1}\right)_{cb},$$

and then the Hamiltonian is

$$H = \sum_{\ell,c,b} \dot{U}_{\ell\, bc} P_{\ell\, cb} - L = \sum_\ell \frac{-g^2}{2a} \mathrm{Tr}\left(U_\ell P_\ell U_\ell P_\ell\right) + V(\{U\}).$$

We notice that this is not the sum of P_ℓ^2's but instead the sum of $U_\ell P_\ell$ squared. So we will define electric field $E_\ell = iU_\ell P_\ell$ in terms of which the Lagrangian is $\propto \mathrm{Tr}\, E^2$.

We will then be a little more carefull, noting that for SU(N) not all elements of U are unconstrained. With greater care we should have taken $U = e^{i\omega^i L_i}$, with N^2-1 fields ω^i as real parameters, and define the quantum momenta as $\Pi_i = -i\frac{\partial}{\partial\omega^i}$. The apparent N^2 elements of E actually take this into account — E gives zero on a phase change of U.

As quantum operators,

$$(E_\ell)_{ab} = i\left(U_\ell P_\ell\right)_{ab} = U_{\ell\, ac}\frac{\partial}{\partial(U_\ell)_{bc}}$$

will turn out to be "left-derivatives" on the gauge group, and the connection of these to Π_i will be quite interesting, especially as it will give us the invariant metric we need for our Lie group to justify the use of the rearrangement theorem.

This is actually the way to formulate hamiltonian mechanics quite generally when the coordinates live in a curved rather than flat Euclidean space.