Last time we found we could make a locally invariant field theory of matter transforming according to a representation of a Lie group, $\Psi(x^\mu) \to e^{-i\boldsymbol{\omega}(x^\mu)}\Psi(x^\mu)$, but only by introducing new vector fields $\mathcal{A}_\nu(x^\mu)$ taking values in the representation of the Lie algebra. This replaced ordinary derivatives of Ψ by *covariant derivatives* $D_\nu\Psi = [\partial_\nu - ig\mathcal{A}_\nu]\,\Psi$. Now we have new degrees of freedom, $\mathcal{A}_\nu(x^\mu)$ which transform under local symmetry (gauge) transformations in an inhomogeneous manner

$$\mathcal{A}'_\nu \to \mathcal{A}'_\nu = e^{-i\boldsymbol{\omega}(x^\mu)}\mathcal{A}_\nu e^{i\boldsymbol{\omega}(x^\mu)} - \frac{i}{e}\left(\frac{\partial}{\partial x_j}e^{-i\boldsymbol{\omega}(x^\mu)}\right)e^{i\boldsymbol{\omega}(x^\mu)}\,.$$

We found we could understand this as the continuum limit of a lattice theory where these gauge degrees of freedom are **group** elements required to understand how to compare Ψ's at different points. Modifying nearest neighbor differences to include parallel transport by the gauge fields, we found we could make the matter part of the Lagrangian locally gauge invariant, but then we needed to ask about finding a group-invariant dynamics for these gauge degrees of freedom. We saw that we could minimize the dependence on local group transformations by considering the placquette, $\mathfrak{G}_P = U_d^{-1}U_c^{-1}U_bU_a$, which is invariant under $\boldsymbol{\omega}(x^\mu)$ at $x^\mu = b$, c, and d, and transforms by $\mathfrak{G}_P \to e^{-i\boldsymbol{\omega}(x_a^\mu)}\mathfrak{G}_P e^{i\boldsymbol{\omega}(x_a^\mu)}$ The remaining gauge dependence could be eliminated by taking the trace in some representation, except that would be a phase rather than a hermitean piece suitable for a lagrangian.

We expect our lattice to be helpful in the continuum limit where Ψ ought to change only slightly between neighboring points, and the parallel transport by one lattice spacing should differ from $\mathbb{I}$ by order the lattice spacing a_ρ, so we expanded $\mathfrak{G}_b = e^{iga^\nu\mathcal{A}_\nu} \sim \mathbb{I} + ia^\nu g\mathcal{A}_\nu - \frac{1}{2}a^{\nu\,2}g^2\mathcal{A}_\nu^2 + \ldots$, and also expressed its x^μ dependance in terms of $\mathcal{A}$ evaluated at the center of the placquette and the derivatives in each direction. We found that $\mathfrak{G}_P \approx 1 + iga^\mu a^\nu\mathcal{F}_{\mu\nu}$ with

$$\mathcal{F}_{\mu\nu}(\mathbf{x}) = \partial_\mu\mathcal{A}_\nu(\mathbf{x}) - \partial_\nu\mathcal{A}_\mu(\mathbf{x}) - ig\left[\mathcal{A}_\mu(\mathbf{x}), \mathcal{A}_\nu(\mathbf{x})\right].$$

We note that this crucial field, the *field strength tensor*, is a representative of the Lie algebra, is an antisymmetric tensor, and transforms under gauge transformations (under the symmetry group) by

$$\mathcal{F}_{\mu\nu}(x^\mu) \to e^{-i\boldsymbol{\omega}(x^\mu)}\mathcal{F}_{\mu\nu}(x^\mu)e^{i\boldsymbol{\omega}(x^\mu)}\,.$$

Today

From the fact that $\operatorname{Tr} G_P$ is gauge invariant, but not hermitean, we are led to the form for the gauge field lagrangian, both for the lattice and for the

continuum. In the continuum, we are led to the covariant derivative operators, and note the connection of their commutators with the field strength tensor.

We will review the meaning of gauge invariance, and some of the ways it complicates normal lagrangian or hamiltonian mechanics. We will go back to the lattice, looking only at the gauge degrees of freedom.

We will take the continuum limit in time, but not in spatial directions, and we will ask about the Lagrangian and Hamiltonian formulations of the dynamics. You may never have explicitly considered dynamics for degrees of freedom which live on curved manifolds rather than Euclidean space, but this will give an enlightening discussion, and also tell us about the manifold of our Lie group.

To get to a Lagrangian, we will get time derivatives of spatially dependent fields by first going to the temporal gauge, where the group elements on all time-links are set to $\mathbb{I}$, and so the time-space placquettes (electric fields) will behave quite differently from the space-space placquettes (magnetic fields). Taking the continuum limit in the time direction, these time-space placquettes will give us a dependence on $\dot{U}$ as well as U, where U is the gauge field on spatial links only. We will concentrate our attention on the $\dot{U}$ terms, as these determine the canonical momenta and thus the transition to the Hamiltonian. We shall see that the canonical momenta do not enter as $\sum_\ell \Pi_\ell^2$, where Π_ℓ is the canonical momentum conjugate to U_ℓ, but rather as the square of "electric field operators", which are "left-derivatives", and do not commute with each other. In terms of the canonical momenta the Hamiltonian involves the metric tensor for the curved group manifold, which we shall see is given in terms of the adjoint representation $[\mathcal{S}(L_a)]_{ij}$ of the generators of the group. The determinant of the metric gives the group-invariant measure, the weight we need to integrate over the group for the rearrangement theorem.

This is actually the way to formulate hamiltonian mechanics quite generally when the coordinates live in a curved rather than flat Euclidean space.