Chapter 12

Tensor Products of Irreducible Representations

Consider two representations with Young Graphs η_1 and η_2, corresponding to tensors of rank k_1 and k_2. The tensor product is a tensor of rank $k_1 + k_2$, and must be decomposed into irreducible representations corresponding to Young Graphs of $S_{k_1+k_2}$. Which ones? I will give only the magic rule, and simultaneously we will do as an example $\yng(2) \times \yng(2,1)$.

1) Take the second graph and place 1's in the first row of boxes, 2's in the second row, *etc.*: $\young(11,2)$.

2) Take the boxes of the top row of the second graph and add them to the first graph in all possible ways resulting in Young graphs. No two boxes may go in the same column.

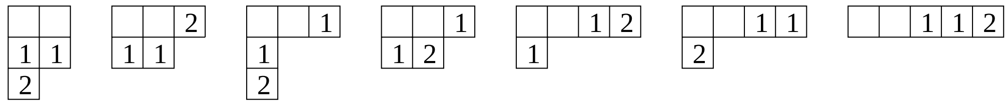
$$\young(\ \ ,11) \qquad \young(\ \ 1,1) \qquad \young(\ \ 11)$$

Successively add the boxes from the lower rows, one row at a time, working down. The configurations are constrained, however, by

(a) After adding each row, the graph must be a Young graph.

(b) No two boxes in a column can have the same number.

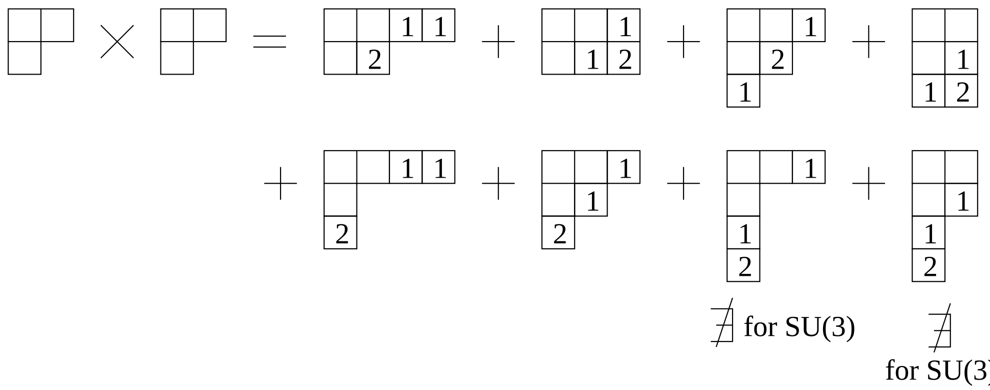
(c) Reading *right-to-left* and top-to-bottom (Hebrew fashion), at any point there must be no more $j+1$'s encountered than j's.



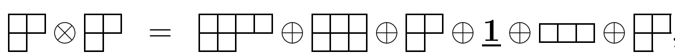
$\overline{\mathbf{3}}$ forbidden $\mathbf{6}$ $\overline{\mathbf{15}}$ forbidden $\mathbf{24}$ forbidden

In SU(3), $\mathbf{6} \times \mathbf{8} = \overline{\mathbf{3}} + \mathbf{6} + \overline{\mathbf{15}} + \mathbf{24}$.

A very important example in SU(3) is octet $\otimes$ octet:



Recall that at most N boxes can be in a column, and a column of N boxes can be dropped, so for SU(3),



$$\text{or} \qquad \mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{8}.$$

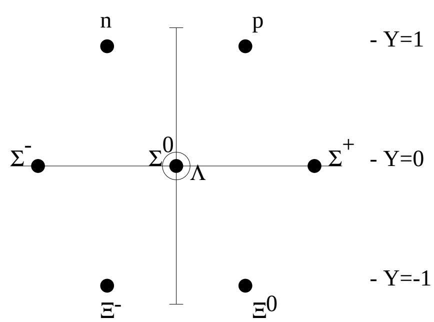
Notice that the octet appears twice in the product of two octets, as each different labelling counts. This complicates the Clebsch-Gordonry. For example, the last time we considered πN scattering, we noted that the S matrix was SU(2) invariant, so we could insert a set of intermediate projections onto states of total isospin T, with amplitudes $a_{1/2}$ and $a_{3/2}$. But now let us generalize to the scattering of any baryon B in the same SU(3) multiple (the octet) as the proton (which means p or n or Λ or $\Sigma^{\pm}$ or Σ^0 or Ξ^0 or Ξ^-) scattering off any meson M in the meson octet: $K, \bar{K}, \pi, \eta$. To the extent that we can ignore the breaking of SU(3) symmetry, we can combine initial two-particle states into states $|\eta, T_3, Y\rangle$ for $\eta = \mathbf{27}, \overline{\mathbf{10}}, \mathbf{10}, \mathbf{1}$, and $\mathbf{8}$, but for the $\mathbf{8}$, there are two ways to combine B's and M's into each state of the intermediate octet. These are called $\mathbf{8}_s$ and $\mathbf{8}_a$. The final state can be similarly combined. Because the scattering matrix S is approximated by being SU(3) invariant, the initial and final states must be in the same SU(3) multiplet. Thus there is an analog of the Wigner-Eckhart theorem which says the scattering amplitude is determined in terms of reduced amplitudes for the **27**, for the **10**, the $\overline{\mathbf{10}}$, and the **1**, together with *four* amplitudes $\mathbf{8}_{ss}$, $\mathbf{8}_{sa}$, $\mathbf{8}_{as}$, and $\mathbf{8}_{aa}$ which describe the four possibilities to combine initial states into a given state in the octet and then let it decompose into the final states.

The classic paper on Clebsch-Gordon coefficients for SU(3) is J. J. DeSwart, *Rev. Mod. Phys.* **35** (1963) 916. We will not go into this subject.

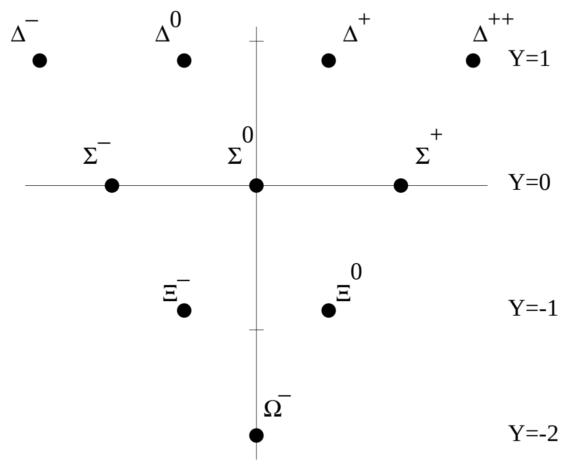
We have described the representations of SU(N) in terms of tensors made of quarks. What are the representations made from antiquarks? More generally, given a representation η, what is its conjugate representation η^*? The answer is found from the requirement that $\Gamma^\eta \times \Gamma^{\eta^*}$ contains the identity representation. As we can loose boxes only by discarding columns of length N, we take our original graph η (say $\ydiagram{4,1}$), and extend it to a rectangle of height N, $\left(\begin{ytableau} X & X & X & X \\ X & + & + & + \\ + & + & + & + \end{ytableau}\text{ for SU(3)}\right)$, discard the original and rotate the rest by 180°, to get η^* $\left(\ydiagram{4,3}\text{ for SU(3)}\right)$. Note that the η^* corresponding to each representation depends on N. For SU(2), each representation is equivalent to its own conjugate, but this is not true for SU(N) for $N > 2$. From this method, we see that $\ydiagram{1,1}$ for SU(3) is the conjugate of the quark $\ydiagram{1}$, so is the antiquark representation. We also see that an initial column of N boxes doesn't affect η^*, so it also doesn't affect η.

Consider a state of three quarks. Under color, the quarks are an SU(3) triplet. Let us consider only the "light flavors" u, d, and s, so they are also an SU(3) triplet under flavor. Finally they are spin $\frac{1}{2}$ fermions, so they are an SU(2) doublet in spin.

In the lowest bound state, the quarks all have the same spatial dependence, in the s state, $\ell = 0$. The total wave function is therefore only dependent on color, flavor, and spin. Fermi statistics requires that overall, the wave function must be antisymmetric. Under flavor, we could have a decuplet $\ydiagram{3}$, an octet $\ydiagram{2,1}$, or a singlet $\ydiagram{1,1,1}$. The same would be true for color, except that by "color confinement" only singlets under color can escape as particles. Thus $\ydiagram{1,1,1}$ is the only acceptible color representation, and is totally antisymmetric. Finally, for SU(2), $\ydiagram{1,1,1}$ doesn't exist, so there are only the $\ydiagram{3}$ $J = 3/2$ and the $\ydiagram{2,1}$ $J = \frac{1}{2}$ possibilities. This must be combined with the flavor representation to give total symmetry (as the color already provides the antisymmetry). From Homework #3 problem 3, $\ydiagram{2,1} \otimes \ydiagram{2,1}$ contains $\ydiagram{3}$, but $\ydiagram{2,1} \otimes \ydiagram{1,1,1}$ and $\ydiagram{2,1} \otimes \ydiagram{3}$ do not. Of course the totally symmetric spin combination and the totally symmetric SU(3) combination have a totally symmetric composite. Thus the quarks can form a decuplet of spin 3/2 particles or an octet of spin 1/2, but nothing else. That this agrees with the lowest mass baryons was the basis of the quark model and the origin of the idea of color $SU(3)$. Without color, we would need flavor $\otimes$ spin to be antisymmetric. This would give an **8** of $J = 1/2$ but a singlet of $J = 3/2$, which is phenomenologically disasterous.



The octet of spin $\frac{1}{2}$ baryons.



The decuplet of spin 3/2 baryons. All except the Ω^- were known when SU(3) (flavor) was proposed, and it predicted the existence (and approximate mass) of the Ω^-, a strong argument for the "eightfold way" assignment of flavor.

I will not consider $O(N)$ and its connection to Clifford algebras and the Dirac equation. The classic reference is
K. M. Case, *Phys. Rev.* **97** (1955) 810.