Chapter 6

SU(3)

SU(3) first hit the Physics world in 1961 through papers by Gell-Mann and Ne'eman which applied it to what we now call the flavor of hadrons, at a time when particles involving charm, top, or bottom were unknown. In modern language, these hadrons are made up of quarks of three different "flavors", called up, down, and strange. The SU(3) acts on these quarks exactly as isospin acts on p's and n's, which is in fact the same as isospin's action on the u and d quarks. That is, the symmetry $\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to U \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, where U is a unitary 3×3 matrix of determinant one, is an approximate symmetry of the Lagrangian or Hamiltonian for strong interactions. The subset of unitary U matrices which leave the third component unchanged is just the isospin group. This **flavor SU(3)** symmetry is, however, clearly not an exact symmetry, even for the strong interactions, as the particles involving strange quarks are considerably heavier than the others.

The interest in SU(3) as a flavor symmetry has largely faded away, after the discovery of 3 more quark flavors which act so differently from the first three that symmetry seems an inappropriate approximation. But in the meantime the theory of QCD, which requires the quarks to have three colors as well as flavors, has become the standard way of understanding the strong interactions, and it is based on an exact **color SU(3)** symmetry, in fact a gauge symmetry. So we still have a good reason to investigate SU(3) in detail, both for its own sake and as an example of more elaborate groups.

The generators of the group are clearly 3×3 hermitean[1] traceless[2] matrices. This is an eight dimensional space, as the 9 real values have the one constraint of tracelessness. The standard basis is Gell-Mann's original one,

$$\lambda_i = \begin{pmatrix} \sigma_i & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for } i = 1,2,3; \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We take the generators to be $T_a = \lambda_a/2$. Note T_1, T_2, and T_3 generate an SU(2) subalgebra which just rotates the isotopic spin doublet $\begin{pmatrix} u \\ d \end{pmatrix}$, leaving the isotopic singlet s invariant, so this is just the isospin group.

The structure constants for the Gell-Mann basis of SU(3) are always called $f_{jk\ell}$, with $[\lambda_j, \lambda_k] = 2if_{jk\ell}\lambda_\ell$. As the first three T_i form an SU(2) subgroup, $f_{jk\ell} = \epsilon_{jk\ell}$ when all indices are ≤ 3, and $f_{jk\ell} = 0$ for $j,k \leq 3$ when $\ell > 3$. Also T_4, T_5 and $\frac{1}{2}\left(\sqrt{3}T_8 + T_3\right)$ form a canonical SU(2) basis [3], and so do T_6, T_7 and $\frac{1}{2}\left(\sqrt{3}T_8 - T_3\right)$, so $f_{458} = \frac{1}{2}\sqrt{3}$, $f_{453} = \frac{1}{2}$, $f_{678} = \frac{1}{2}\sqrt{3}$, $f_{673} = -\frac{1}{2}$. Finally $[\lambda_1, \lambda_4] = i\lambda_7$, so $f_{147} = \frac{1}{2}$, and similarly $f_{246} = f_{257} = \frac{1}{2}$, $f_{156} = -\frac{1}{2}$. The others are given by total antisymmetry, or vanish.

The commutators are a general feature of a Lie algebra, but for SU(n) only there is also an anticommutator relation,

$$\{\lambda_i, \lambda_j\} = \frac{4}{3}\delta_{ij}\mathbb{1} + 2d_{ijk}\lambda_k$$

because the λ's and $\mathbb{1}$ are a complete set of hermitean matrices.

The matrices are themselves a representation (the **defining representation**, *not* the adjoint representation). There can only be 2 independent

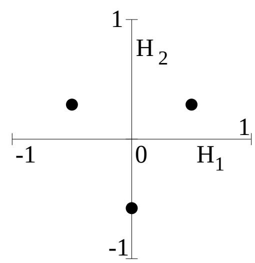
[1] An $N \times N$ hermitean matrix H is diagonalizable with N real eigenvalues λ_j, so the exponential e^{iH} has the same eigenvectors with eigenvalues $e^{i\lambda_j}$, which has magnitude 1, so e^{iH} is unitary.

[2] Quite generally for a finite square matrix M, $\det M = \exp \mathrm{Tr}(\ln M)$.

[3] One often defines $\sqrt{2}\; U_\pm = T_6 \pm iT_7$ and $\sqrt{2}\; V_\pm = T_4 \pm iT_5$, $2U_3 = \sqrt{3}T_8 - T_3$ $2V_3 = \sqrt{3}T_8 + T_3$, so the U's form an SU(2), and so do the V's. Finally $Y = 2T_8/\sqrt{3}$ is called hypercharge.

simultaneously diagonalized traceless hermitean 3×3 matrices, and we already have them in λ_3 and λ_8, so the Cartan subalgebra is $\mathcal{H} = \langle T_3, T_8 \rangle$. Let $H_1 = T_3$, $H_2 = T_8$.

The weights in this representation are just the combinations $(H_{1\,ii}, H_{2\,ii})$ (no sum) or $\frac{1}{2}(\lambda_{3\,ii}, \lambda_{8\,ii})$ (no sum) or



$$(\mu_1, \mu_2) = \underbrace{\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)}_{u}, \quad \underbrace{\left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)}_{d}, \quad \underbrace{\left(0, -\frac{1}{\sqrt{3}}\right)}_{s}.$$

The three root vectors E_α and their conjugates $E_{-\alpha}$ span the six dimensions orthogonal to $\mathcal{H}$. These make transitions between the weights. The root which takes $d \to u$, which we used to call T^+, is now $E_{(1,0)} = \frac{1}{\sqrt{2}}(T_1 + iT_2)$. We also have $E_{(1/2,\sqrt{3}/2)} = \frac{1}{\sqrt{2}}(T_4 + iT_5)$, sometimes called V_+, which takes $s \to u$, and $E_{(-1/2,\sqrt{3}/2)} = \frac{1}{\sqrt{2}}(T_6 + iT_7)$, also called U_+, which takes $s \to d$.

So the root space of the generators looks like the figure, with the roots forming a regular hexagon, with angles between them of $60° \times n$. From the diagram we see T_- generates doublets on V_+ and U_- as required by $\dfrac{\vec{\alpha}_{T_+} \cdot \vec{\alpha}_{V_+}}{\vec{\alpha}^{\,2}_{T_+}} = \dfrac{n}{2} = \dfrac{1}{2}$. Similarly U_- generates doublets ("U spin doublets") starting with V_+ or T_-.