Chapter 4

SU(2)

4.1 Representations of SU(2)

We will now work out in detail the properties of SU(2) and its representations. We have already seen that the generators may be chosen to be

$$L_i = \frac{1}{2}\sigma_i, \qquad \text{with } \sigma_i = \text{the Pauli matrices.}$$

Then $c_{ij}{}^k = \epsilon_{ijk}$ are the structure constants, and

$$\beta_{ij} = -\epsilon_{aib}\epsilon_{bja} = 2\delta_{ij}.$$

Thus our generators are not quite canonically normalized, but are all normalized equally, and β is positive definite. This is related to the fact, which we have already seen, that the group is compact.

In writing down our generators, we have chosen, arbitrarily, one direction to make diagonal. Any rotation can be, by similarity transformation with at rotation, rotated into the z direction, so as we have a choice as to which of the infinite number of equivalent representations to choose, we may choose L_3 to be diagonal.

If we had several generators which commuted with each other, we could have chosen all of them to be diagonal. In general, we will take a maximal set of commuting generators, called the **Cartan subalgebra**, and represent them as diagonal.

For SU(2), no rotations about any other axis commute with L_3, so the Cartan subalgebra is one dimensional.

We want to look for finite dimensional irreducible representations, and we have chosen to make $\Gamma(L_3)$ diagonal. Consider a basis vector e_m with $L_3 e_m = m e_m$, or $\Gamma_{m'm}(L_3) = m\delta_{m'm}$, where so far we are not restricting m other than to say it is real. However, we know that $e^{4\pi i L_3} = \mathbb{1}$, so $e^{4\pi i m} = 1$, and $m = n/2$ for some integer n, for us to have a true representation of the SU(2) group. But even without assuming this, we will find it anyway, from requiring the representation to be finite dimensional.

From L_1 and L_2 we can form the two operators

$$L_\pm = \frac{L_1 \pm iL_2}{\sqrt{2}}.$$

These are not actually part of the Lie algebra, because that is defined over the reals, but any representation of the algebra is automatically a representation of the complexification, *i.e.* $\{\sum v_a L_a\}, v_a \in \mathbb{C}$.

$L_\pm$ are called raising and lowering operators respectively because

$$[L_3, L_\pm] = \frac{1}{\sqrt{2}}[L_3, L_1] \pm \frac{i}{\sqrt{2}}[L_3, L_2] = \frac{i}{\sqrt{2}}L_2 \pm \frac{1}{\sqrt{2}}L_1 = \pm L_\pm.$$

Thus if $L_3 e_m = m e_m$,

$$L_3(L_\pm e_m) = (L_\pm L_3 \pm L_\pm)e_m = (m \pm 1)L_\pm e_m,$$

so $L_\pm e_m$ is a new basis vector of the representation, unless it vanishes.

Applying L_+ an arbitrary number of times generates an arbitrary number of vectors unless at some point it gives 0. All of these vectors are linearly independent, so we would generate an infinite-dimensional space unless there exists some state $e_j \propto L_+^p e_m$ with $L_3 e_j = j e_j$ on which $L_+ e_j = 0$. e_j is called the **highest weight state**.

Let us form a normalized vector proportional to e_j called[1] $|j, j\rangle$, and write the inner product in quantum mechanical form

$$\langle j, j | j, j \rangle = 1.$$

Now we generate a sequence of orthonormal states

$$|j, m\rangle = N_{j,m} L_-^{j-m} |j, j\rangle,$$

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[1] If there are several eigenvectors with eigenvalue m, add a label α to the states generated by L_+, so the highest weight ones will be $|j, j, \alpha\rangle$, but the states with different α will turn out to be in different irreducible representations, as shown below.

where $N_{j,m}$ is a real normalization constant not quite the same as Georgi's. The state $|j, m\rangle$ is an eigenstate of L_3 with eigenvalue m,

$$L_3 |j, m\rangle = m |j, m\rangle$$

because each application of L_- lowers the eigenvalue of L_3 by one[2]. The normalization factors can be found by observing that

$$|j, m\rangle = L_- |j, m + 1\rangle \underbrace{\frac{N_{j,m}}{N_{j,m+1}}}_{N_{m+1}^{-1} \text{ in Georgi}}$$

or $L_- |j, m + 1\rangle = N_{m+1} |j, m\rangle$.

On the other hand,

$$
\begin{aligned}
L_+ |j, m\rangle &= N_{j,m} L_+ L_-^k |j, j\rangle \quad \text{with } k = j - m \\
&= N_{j,m} \left(\sum_{r=0}^{k-1} L_-^{k-1-r} [L_+, L_-] L_-^r |j, j\rangle + L_-^k L_+ |j, j\rangle \right).
\end{aligned}
$$

But $[L_+, L_-] = \frac{1}{2} [L_1 + iL_2, L_1 - iL_2] = i [L_2, L_1] = L_3$ and $L_3 L_-^r |j, j\rangle = (j - r) L_-^r |j, j\rangle$ as L_- is the lowering operator. So as $L_+ |j, j\rangle = 0$,

$$
\begin{aligned}
L_+ |j, m\rangle &= N_{j,m} \sum_{r=0}^{k-1} (j - r) L_-^{k-1} |j, j\rangle \\
&= \left[kj - \frac{k(k-1)}{2} \right] N_{j,m} L_-^{k-1} |j, j\rangle \\
&= k \left[j - \frac{k-1}{2} \right] N_{j,m} N_{j,m+1}^{-1} |j, m + 1\rangle \\
&= \frac{1}{2} (j - m)(j + m + 1) N_{m+1}^{-1} |j, m + 1\rangle .
\end{aligned}
$$

Now $L_+ = L_-^\dagger$, so

$$
\begin{aligned}
\langle j, m + 1 | L_+ | j, m\rangle &= \langle j, m | L_- | j, m + 1\rangle^* \\
= & \qquad = \\
\frac{1}{2} (j - m)(j + m + 1) N_{m+1}^{-1} &= \qquad N_{m+1}
\end{aligned}
$$

[2] The state $|j, m, \alpha\rangle$ so generated will be proportional to the original e_m because $[L_-, L_+] = -L_3$ and e_m is an eigenstate of L_3. Thus all the states generated have the same α.

or

$$
\begin{aligned}
N_{m+1} &= \frac{1}{\sqrt{2}} \sqrt{(j - m)(j + m + 1)} \\
N_m &= \frac{1}{\sqrt{2}} \sqrt{(j - m + 1)(j + m)}.
\end{aligned}
$$

Now the set of states $|j, m\rangle$ $m = j, j - 1, j - 2, \dots$ must terminate somewhere if the representation is to be finite dimensional, so for some $m < j$, $N_m = 0$, so $j + m = 0$, or $m = -j$. But $j - m$ is an integer, so $2j$ is an integer, and hence $2m$ is an integer as well.

We have formed a representation with orthonormal basis $\{|j, m\rangle, m = j, j - 1, j - 2, \dots, -j\}$, which is $2j + 1$ dimensional, with $2j$ an integer. We found the representation of L_3 and $L_\pm$, from which L_1 and L_2 follow[3].

There is exactly one irreducible representation of SU(2) for each dimension and for each $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. Each representation has just one eigenstate of L_3 with each eigenvalue $m \in [-j, -j + 1, \dots, j]$. The representation is given by

$$
\begin{aligned}
L_3 |j, m\rangle &= m |j, m\rangle \\
L_+ |j, m\rangle &= \frac{1}{\sqrt{2}} \sqrt{(j - m)(j + m + 1)} |j, m+1\rangle \\
L_- |j, m\rangle &= \frac{1}{\sqrt{2}} \sqrt{(j + m)(j - m + 1)} |j, m-1\rangle
\end{aligned}
$$

the representation of the group elements is, of course, given by the exponential of the representation of the generators. The $2j + 1$ dimensional representation is usually denoted

$$\mathcal{D}^j_{m_1, m_2}(g)$$

The $j = \frac{1}{2}$ representation is just the group elements themselves. Therefore it is called the **defining representation**.

[3] Notice that if we started with 2 orthogonal states $e_{m,\alpha}$ and $e_{m,\beta}$, all the states we generated from $e_{m,\alpha}$ would be orthogonal to all those from $e_{m,\beta}$, and the first set by itself would be an irreducible representation.

4.2 Reduction of Direct Products

If Γ^1 and Γ^2 are two representations of a Lie group, and Γ is their direct product, then

$$\Gamma_{(ix),(jy)}(g) = \Gamma^1_{ij}(g)\Gamma^2_{xy}(g).$$

Recalling that[4] $\Gamma(L_a) = -i\,\frac{d}{dv_a}\Gamma(g(v_a))\big|_{\substack{v=0\\g=1}}$,

$$\Gamma_{(ix),(jy)}(L) = \Gamma^1_{ij}(L)\Gamma^2_{xy}(\mathbb{1}) + \Gamma^1_{ij}(\mathbb{1})\Gamma^2_{xy}(L) = \delta_{xy}\Gamma^1_{ij}(L) + \delta_{ij}\Gamma^2_{xy}(L).$$

Now for SU(2), consider the direct product states with bases

$$|j_1, j_2; m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$

This representation can in principle be decomposed into a set of irreducible representations $|j, m\rangle$. To see which, we count the dimensionality of the eigenspaces of L_3,

$$
\begin{aligned}
L_3 |j_1, j_2; m_1, m_2\rangle &= (L_3 |j_1, m_1\rangle) \otimes |j_2, m_2\rangle + |j_1, m_1\rangle \otimes (L_3 |j_2, m_2\rangle)\\
&= (m_1 + m_2) |j_1, m_1; j_2, m_2\rangle.
\end{aligned}
$$

Fortunately our states are already eigenstates of L_3. The largest eigenvalue is $j_1 + j_2$, which can occur in only one way. So the representation $j = j_1 + j_2$ occurs exactly once in the direct product. This representation accounts for one of the basis vectors for each m (= eigenvalue of L_3) with $|m| \le j_1 + j_2$.

Now consider $m = m_1 + m_2 = j_1 + j_2 - 1$. There are two ways to build it up, with $m_1 = j_1 - 1, m_2 = j_2$, or with $m_1 = j_1, m_2 = j_2 - 1$, as long as each j is at least $\frac{1}{2}$. The next higher m, $m = j_1 + j_2 - 2$, can be built three different ways. The growth stops with $m = j_1 + j_2 - k$, which can be written in $k + 1$ ways, with $m_1 = j_1 - r, m_2 = j_2 - k + r, r = 0, \ldots, k$. What stops it is that we must require $m_1 \ge -j_1$, and $m_2 \ge -j_2$, so $k \le 2\min(j_1, j_2)$. After that, the m corresponding to the smaller j (say $j_{\min} = k/2$)) takes on all $2j_{\min} + 1$ values, until we get to $m < -|j_{\max} - j_{\min}|$, where the possible values of m for the smaller j are limited.

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[4]I have added the $-i$ to get physicist's generators.

Example:

$$j_1 = 5/2$$
$$j_2 = 1$$

Thus the dimensionality of the m subspace is $\min(j_1 + j_2 - |m| + 1, 2j_{\min} + 1)$.

m	ways $m = m_1 + m_2$	number
7/2	$5/2 + 1$	1
5/2	$5/2 + 0;\ 3/2+1$	2
3/2	$5/2 + (-1);\ 3/2+0;\ 1/2+1$	3
1/2	$3/2 + (-1);\ 1/2+0;\ -1/2+1$	3
$-1/2$	$1/2 + (-1);\ -1/2+0;\ -3/2+1$	3
$-3/2$	$-1/2 + (-1);\ -3/2+0;\ -5/2+1$	3
$-5/2$	$-3/2 + (-1);\ -5/2+0$	2
$-7/2$	$-5/2 + (-1)$	1

Each decrease of m gives a new state, which is the highest state of a new representation, up to the point where

$$j_1 + j_2 - m + 1 = 2j_{\min} + 1,$$

or $|m| = j_{\max} - j_{\min} = |j_1 - j_2|$. Thus

$$\Gamma^{j_1} \otimes \Gamma^{j_2} \cong \bigoplus_{j=|j_1 - j_2|}^{j_1+j_2} \Gamma^j.$$

This statement is an equivalence, so there must be a unitary matrix U which connects the direct product to the direct sum. This matrix has a right index which is a pair (m_1, m_2) and a left index which must specify which irreducible representation in the sum is referenced, and which m for that representation. Because each j in the sum appears only once, we can use j to index the representation. Thus the left index is (j, m). Of course the whole matrix depends on j_1 and j_2. It is generally written[5]

$$(j_1, j_2, j, m| j_1, j_2, m_1, m_2)$$

with hermitean conjugate $(j_1, j_2, m_1, m_2| j_1, j_2, j, m)$ and with

$$|j, m\rangle = \sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle (j_1, j_2, m_1, m_2| j_1, j_2, j, m).$$

Both basis states are normalized, so $(j_1, j_2, m_1, m_2| j_1, j_2, j, m)$ is a unitary matrix. That does not completely define it, of course, because each of

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[5]Notice that when numbers are inserted, it is ambiguous whether the third index is the total j or the first m.

the invariant subspaces could be multiplied by an arbitrary phase $|j, m\rangle \rightarrow e^{i\phi_j} |j, m\rangle$. For each j, we choose reality and the sign convention by making this overlap for $m = j$ and $m_1 = j_1$ be real and positive:

$$\text{SC:} \qquad (j_1, j_2, j, m = j| j_1, j_2, m_1 = j_1, m_2 = j - j_1) > 0.$$

A detailed example:

Consider the direct product of $j_1 = 1$ and $j_2 = \frac{1}{2}$, as occurs if we ask about the total angular momentum of an electron (spin $\frac{1}{2}$) in a p orbital (orbital angular momentum 1, in units of $\hbar$).

$$|j_1 = 1, m_1 = 1\rangle \otimes \left|j_2 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle,$$

so, abbreviating $(j_1, j_2, m_1, m_2| j_1, j_2, j, m)$ as $(j_1, j_2, m_1, m_2| j, m)$, we have $\left(1, \frac{1}{2}, 1, \frac{1}{2}\big| \frac{3}{2}, \frac{3}{2}\right) = 1$. Applying the lowering operator

$$L_- \left|\frac{3}{2}, \frac{3}{2}\right\rangle = (L_- |1, 1\rangle) \otimes \left|\frac{1}{2}, \frac{1}{2}\right\rangle + |1, 1\rangle \otimes \left(L_- \left|\frac{1}{2}, \frac{1}{2}\right\rangle\right)$$

$$\underset{=}{} \qquad \underset{=}{}$$

$$\sqrt{\frac{1}{2} \cdot 3 \cdot 1} \left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{1}{2} \cdot 2 \cdot 1} \left|1, 0, \frac{1}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{1}{2} \cdot 1 \cdot 1} \left|1, 1, \frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$\text{or} \qquad \left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} \left|1, 0, \frac{1}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}} \left|1, 1, \frac{1}{2}, -\frac{1}{2}\right\rangle$$

so $\left(1, \frac{1}{2}, 0, \frac{1}{2}\big| \frac{3}{2}, \frac{1}{2}\right) = \sqrt{\frac{2}{3}}$; $\quad \left(1, \frac{1}{2}, 1, -\frac{1}{2}\big| \frac{3}{2}, \frac{1}{2}\right) = \sqrt{\frac{1}{3}}$.

The state $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$ is orthogonal to $\left|\frac{3}{2}, \frac{1}{2}\right\rangle$ but also must be made of the 2 $m_1 + m_2 = \frac{1}{2}$ states, so

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \alpha \left|1, \frac{1}{2}, 0, \frac{1}{2}\right\rangle + \beta \left|1, \frac{1}{2}, 1, -\frac{1}{2}\right\rangle.$$

Orthogonality tells us $\sqrt{2/3}\, \alpha + \sqrt{1/3}\, \beta = 0$ and normalization that $\alpha^2 + \beta^2 = 1$. The sign convention SC says $\beta > 0$, so

$$\beta = \sqrt{\frac{2}{3}} = \left\langle\frac{1}{2}, \frac{1}{2}\big\| 1, \frac{1}{2}, 1, -\frac{1}{2}\right\rangle$$

$$\alpha = -\sqrt{\frac{1}{3}} = \left\langle\frac{1}{2}, \frac{1}{2}\big\| 1, \frac{1}{2}, 0, \frac{1}{2}\right\rangle$$

Applying L_- further to the states $\left|\frac{3}{2}, m\right\rangle$ and $\left|\frac{1}{2}, m\right\rangle$ specifies the remaining coefficients without additional sign conventions.

Exercise: work out $\left\langle 1, \frac{1}{2}, j, m\big\| 1, \frac{1}{2}, m_1, m_2\right\rangle$ for all nonzero elements.

These orthogonal matrices are called the **Vector coupling coefficients**. They are more usually called, by physicists and incorrectly, **Clebsch-Gordon coefficients**.

The decomposition of the direct product of irreducible representations of a compact group can, of course, always be done, in a fashion analogous to this, as we shall see later. There are, however, special features of SU(2) related to

1. Each representation j appears at most once in $j_1 \otimes j_2$, and all representations of a given dimension are equivalent. This is not true for SU(3), for example.

2. Because there is only one representation for each dimension, each representation is self-conjugate, $\Gamma^{j*} \cong \Gamma^j$. As a consequence, it is possible to treat $j_1 \otimes j_2 \rightarrow j_3$ in a symmetric fashion, defining the Wigner coefficient

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

There is a great machinery for dealing with SU(2) representations, with such objects as 6-j and 9-j symbols, *etc.*. These are used in considering the overlaps of differing choices in how to combine the many angular momenta of components in an atom or in an atomic nucleus. References are Edmunds, Rose, and also Yutsis[6]. To see how crazy one can get, see J. Shapiro, Comp. Phys. Comm. **1**, (69) 207.

3. Starting with the defining representation, $j = \frac{1}{2}$, one can generate an arbitrary representation of spin j by taking the totally symmetric piece of the direct product of $2j$ defining representations $\frac{1}{2} \otimes \frac{1}{2} \otimes \ldots \otimes \frac{1}{2}$. Writing $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$ as $\uparrow$ and $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle$ as $\downarrow$, one can write such an expression, as, for example

$$|2, 1\rangle = \frac{1}{2} (\uparrow\uparrow\uparrow\downarrow + \uparrow\uparrow\downarrow\uparrow + \uparrow\downarrow\uparrow\uparrow + \downarrow\uparrow\uparrow\uparrow).$$

[6]A. P. Yutsis, I. B. Levinson and V. V. Vanagas, "The Theory of Angular Momentum" (Oldbourn Press, London, 1962).

4.3   Representations of Finite Rotations

Consider the action of a finite rotation on an irreducible representation

$$e^{i\nu L}|j,m,\alpha\rangle = \sum_{m'} \mathcal{D}^j_{m',m}\left(e^{i\nu L}\right)|j,m',\alpha\rangle,$$

where α is some index describing all the other features of the states, not having to do with rotations, *e.g.* the principle quantum number.

Then $\langle j,m',\alpha|\, e^{i\nu L}\,|j,m,\alpha\rangle = \mathcal{D}^j_{m',m}(g),$ where $g = e^{i\nu L}.$

Now a direct product state $|j_1,m_1,\alpha_1\rangle \otimes |j_2,m_2,\alpha_2\rangle$ can be decomposed into irreducible representations
$= |j_1,j_2,j,m,\alpha_1,\alpha_2\rangle \langle j_1,j_2,j,m|j_1,j_2,m_1,m_2\rangle$

so $\langle j_1,m'_1,\alpha_1| \otimes \langle j_2,m'_2,\alpha_2|\, e^{i\nu L}\,|j_1,m_1,\alpha_1\rangle \otimes |j_2,m_2,\alpha_2\rangle$
$$= \mathcal{D}^{j_1}_{m'_1,m_1}(g)\mathcal{D}^{j_2}_{m'_2,m_2}(g) \tag{4.1}$$
$$= \sum_{j,m,m'} \langle j_1,j_2,m'_1,m'_2|j_1,j_2,j,m\rangle \mathcal{D}^j_{m',m}(g)\langle j_1,j_2,j,m|j_1,j_2,m_1,m_2\rangle$$

We have not yet shown that there exists an invariant integration measure on the group, and hence that we can prove a great orthogonality theorem. We shall do so later, but for now let us assume it

$$\int dg\, \mathcal{D}^{j_1\,*}_{m'_1,m_1}(g)\, \mathcal{D}^{j_2}_{m'_2,m_2}(g) = N_{j_1}\delta_{j_1,j_2}\delta_{m_1,m_2}\delta_{m'_1,m'_2}. \tag{4.2}$$

To get the normalization correct, normalize $\int dg\, 1 = 1$. Now $\mathcal{D}$ is a unitary matrix, so

$$\sum_m \mathcal{D}^{j\,*}_{m,\mu}(g)\, \mathcal{D}^j_{m,\nu}(g) = \delta_{\mu\nu},$$

so integrating the left hand side gives

$$\delta_{\mu\nu} = N_j\delta_{\mu\nu}\underbrace{\sum_m \delta_{m,m}}_{2j+1}$$

so $N_j = \dfrac{1}{2j+1}$.   [Compare this to $\dfrac{g}{l_i}$ for the finite group representations.]

Applying this to (4.1)

$$\int dg\, \mathcal{D}^{j_3\,*}_{m'_3,m_3}(g)\mathcal{D}^{j_1}_{m'_1,m_1}(g)\mathcal{D}^{j_2}_{m'_2,m_2}(g)$$
$$= \sum_{j,m,m'} \langle j_1,j_2,m'_1,m'_2|j_1,j_2,j,m'\rangle\langle j_1,j_2,j,m|j_1,j_2,m_1,m_2\rangle\frac{\delta_{j,j_3}\delta_{m,m_3}\delta_{m',m'_3}}{2j+1}$$
$$= \frac{1}{2j+1}\langle j_1,j_2,m'_1,m'_2|j_1,j_2,j_3,m'_3\rangle\langle j_1,j_2,j_3,m_3|j_1,j_2,m_1,m_2\rangle \tag{4.3}$$

In atomic or nuclear physics, one very often wants to calculate the amplitude for emission of a photon from some excited state. The photon emission operator is proportional to $e^{i\vec{k}\cdot\vec{r}}$, where $\vec{k}$ is the momentum of the emitted photon and $\vec{r}$ is the position operator acting on the constituents. In fact, for nonrelativistic particles it is exactly

$$\langle\psi_f|\, e^{i\vec{k}\cdot\vec{r}}\,|\psi_i\rangle$$

which determines the transition amplitude. Generally the transition energy corresponds to a photon momentum which is much less than the inverse size of the system, so $|\vec{k}\cdot\vec{r}| \ll 1$ whereever $\vec{r}$ has a significant matrix element. So we can expand in a multipole expansion

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi\sum_{\ell,m} i^\ell Y^{m\,*}_\ell(\hat{k})Y^m_\ell(\hat{r})j_\ell(|k||r|).$$

As the $j_\ell(kr) \propto (kr)^\ell$ for small kr, we can expect this expansion, plugged into our matrix element, to give a series of rapidly converging terms. The spherical harmonics Y^m_ℓ, for a given ℓ, are a set of functions which transform under the spin ℓ representation of SU(2).

Consider a rotation of the particles $\vec{r}$ and their spins, but not the photon. The states ψ_i and ψ_f are eigenstates of a spherically symmetric Hamiltonian so they have definite total angular momentum j_i and j_f, and z components m_i and m_f, along with other quantum numbers α_i and α_f. So with $|\psi_i\rangle = |j_i,m_i,\alpha_i\rangle$, if we consider a rotation $e^{i\nu L}$, we have

$$|j_i,m_i,\alpha_i\rangle \to e^{i\nu L}|j_i,m_i,\alpha_i\rangle = \sum_{m'_i} |j_i,m'_i,\alpha_i\rangle\, \mathcal{D}^{j_i}_{m'_i,m_i}\left(e^{i\nu L}\right)$$
$$\langle j_f,m_f,\alpha_f| \to \langle j_f,m_f,\alpha_f|\, e^{-i\nu L} = \sum_{m'_f} \mathcal{D}^{j_f\,*}_{m'_f,m_f}\left(e^{i\nu L}\right)\langle j_f,m'_f,\alpha_f|$$
$$e^{i\nu L}Y_{\ell m}(\hat{r})e^{-i\nu L} = \mathcal{D}^\ell_{m',m}\left(e^{i\nu L}\right)Y_{\ell m'}(\hat{r}).$$

All taken together makes for no change, so

$$\langle j_f, m_f, \alpha_f | Y_{\ell m}(\hat{r}) | j_i, m_i, \alpha_i \rangle$$
$$= \sum_{m' \, m_i' \, m_f'} \mathcal{D}^{j_f \, *}_{m_f', m_f} \mathcal{D}^{\ell}_{m', m} \mathcal{D}^{j_i}_{m_i', m_i} \langle j_f, m_f', \alpha_f | Y_{\ell m'} | j_i, m_i', \alpha_i \rangle$$

for all ν. Integrating over the invariant measure of the group,

$$\langle j_f, m_f, \alpha_f | Y_{\ell m}(\hat{r}) | j_i, m_i, \alpha_i \rangle = \langle \ell j_i j_f m_f | \ell j_i m m_i \rangle \langle j_f \alpha_f || Y || j_i \alpha_i \rangle,$$

where

$$\langle j_f \alpha_f || Y || j_i \alpha_i \rangle := \frac{1}{2j_f + 1} \sum_{m' \, m_i' \, m_f'} \langle \ell j_i m' m_i' | \ell j_i j_f m_f' \rangle \langle j_f, m_f', \alpha_f | Y_{\ell m'} | j_i, m_i', \alpha_i \rangle.$$

This statement is called the **Wigner-Eckhart theorem**. It is a nice division of the matrix elements into a piece which is pure group theory, the Clebsch-Gordon coefficient, and a piece which is more nearly pure physics, $\langle j_f \alpha_f \| Y \| j_i \alpha_i \rangle$. It is generally written for a set of operators $\mathcal{O}^{\ell}_m$ which transform like Y^{ℓ}_m, for which it is written

$$\langle j_f, m_f, \alpha_f | \mathcal{O}^{\ell}_m | j_i, m_i, \alpha_i \rangle = \langle \ell j_i j_f m_f | \ell j_i m m_i \rangle \langle j_f \alpha_f \| \mathcal{O}^{\ell} \| j_i \alpha_i \rangle,$$

The last factor is called the **reduced matrix element**.

4.3.1   Isospin

In elementary particle physics the number of particles is not conserved due to the possibility of pair creation. For that reason one cannot work with N particle wave functions, but rather with a Hilbert space which contains subspaces having differing numbers of particles. The various subspaces can be built up from the "vacuum" state $|0\rangle$ by applying operators which create particles in a certain state. Let $P^{\dagger}_{\alpha}$ be the operators which creates a proton with properties specified by α (which includes momentum and spin value). Then $P^{\dagger}_{\alpha} |0\rangle$ is a single proton state, while $P^{\dagger}_{\alpha} P^{\dagger}_{\beta} |0\rangle$ is a state which has two protons, one with properties α and one with β. This corresponds to the same physical state as $P^{\dagger}_{\beta} P^{\dagger}_{\alpha} |0\rangle$ though there could be an arbitary phase difference.

Because they are fermions, states are "antisymmetric under interchange", which we insure by taking $P^{\dagger}_{\alpha} P^{\dagger}_{\beta} = -P^{\dagger}_{\beta} P^{\dagger}_{\alpha}$. This automatically gives $(P^{\dagger}_{\alpha})^2 = 0$, or the Pauli exclusion principle.

The hermitean conjugate operator P_{α} annihilates a proton in the state α, if it exists, or else it annihilates the *state*. The algebra of the P's is

$$\{P_{\alpha}, P_{\beta}\} = \left\{P^{\dagger}_{\alpha}, P^{\dagger}_{\beta}\right\} = 0, \qquad \left\{P_{\alpha}, P^{\dagger}_{\beta}\right\} = \delta_{\alpha\beta}.$$

A similar set of operators exist to create neutrons, $N^{\dagger}_{\alpha}$, or electrons $E^{\dagger}_{\alpha}$.

A state consisting on one proton and one neutron can be described by $P^{\dagger}_{\alpha} N^{\dagger}_{\beta} |0\rangle$ or by $N^{\dagger}_{\beta} P^{\dagger}_{\alpha} |0\rangle$. These can differ by a phase, which is really a matter of convenience. We choose anticommuting operators $\left\{P^{\dagger}_{\alpha}, N^{\dagger}_{\beta}\right\} = 0$ for all different fermion fields. Representing different particles, the P and N operators must have trivial commutation relations, and as the creation operators have been chosen to anticommute, we also need $\{P^{\dagger}_{\alpha}, N_{\beta}\} = 0$.

Now consider the operator

$$\sum_{\alpha} P^{\dagger}_{\alpha} N_{\alpha}.$$

It looks around for a neutron to annihilate and, when it finds one, replaces it with a proton of the same momentum and spin. This is not a "physical" operator in the sense that there is no way one can do just this, but it is a mathematical object with an interesting consequence.

Before we consider this further, note that $\sum_{\alpha} P^{\dagger}_{\alpha} P_{\alpha}$ looks around for protons it can annihilate, does so but then recreates it and adds a notch to its belt, so in the end it does nothing but count the number of protons in the state.

In nuclear physics, the bulk of the Hamiltonian is the "strong" or nuclear interactions, which seem to treat the protons and neutrons as equivalent. Of course protons and neutrons have different charges, so the electromagnetic (and electroweak) interactions do distinguish between them. Let us write the hamiltonian as $H_S + H_{EW}$ and write the fact that H_S does not distinguish between protons and neutrons by

$$\left[H_S, \sum_{\alpha} P^{\dagger}_{\alpha} N_{\alpha}\right] = 0.$$

This is because it doesn't matter if you first replace a neutron by a proton and then let H_S act, or let H_S act first and then do the replacement, as H_S acts the same way on both particles. The hermetian conjugate gives $\left[H_S, \sum N^{\dagger}_{\alpha} P_{\alpha}\right] = 0$.

Let's forget H_{EW} for now, and consider

$$T^+ = \frac{1}{\sqrt{2}} \sum_\alpha P_\alpha^\dagger N_\alpha$$

so $\quad [H, T^+] = [H, T^-] = 0$

$$T^- = \frac{1}{\sqrt{2}} \sum_\alpha N_\alpha^\dagger P_\alpha$$

Let

$$
\begin{aligned}
T_3 &= [T^+, T^-] = \frac{1}{2} \sum_{\alpha\beta} \left[P_\alpha^\dagger N_\alpha, N_\beta^\dagger P_\beta \right] = \frac{1}{2} \sum_{\alpha\beta} \left(P_\alpha^\dagger N_\alpha N_\beta^\dagger P_\beta - N_\beta^\dagger P_\beta P_\alpha^\dagger N_\alpha \right) \\
&= \frac{1}{2} \sum_{\alpha\beta} \left(P_\alpha^\dagger \left\{ N_\alpha, N_\beta^\dagger \right\} P_\beta - \left\{ P_\alpha^\dagger, N_\beta^\dagger \right\} N_\alpha P_\beta + N_\beta^\dagger P_\alpha^\dagger \left\{ N_\alpha, P_\beta \right\} \right. \\
&\qquad\qquad \left. - N_\beta^\dagger \left\{ P_\alpha^\dagger, P_\beta \right\} N_\alpha \right) \\
&= \frac{1}{2} \sum_{\alpha\beta} \left(P_\alpha^\dagger \delta_{\alpha\beta} P_\beta - 0 + 0 - N_\beta^\dagger \delta_{\alpha\beta} N_\beta \right) \\
&= \frac{1}{2} \sum_\alpha \left(P_\alpha^\dagger P_\alpha - N_\alpha^\dagger N_\alpha \right).
\end{aligned}
$$

We see that $2T_3$ counts the number of protons minus the number of neutrons.

As

$$[H, T_3] = \left[H, [T^+, T^-] \right] = - \left[T^+, [T^-, H] \right] - \left[T^-, [H, T^+] \right]$$

by the Jacobi identity, and as both terms vanish because $[H, T^\pm] = 0$, we have $[H, T_3] = 0$. One can also show, as for angular momentum, that $[T_3, T^\pm] = \pm T^\pm$, so the T's (or more precisely T_3, $T_1 = (T^+ + T^-)/\sqrt{2}$, and $T_2 = (T^+ - T^-)/\sqrt{2}\, i$) are the generators of an SU(2) symmetry group under which the strong interaction hamiltonian is invariant. Note that T^+ increases the charge by $1|e|$, so a multiplet consists of a sequence of charges differing by e.

To the extent that H_{EW} can be ignored, states should form multiplets (irreducible representations) of this isotopic spin symmetry. For example:

(p, n)	$T = \frac{1}{2}$
deuteron	$T = 0$
H^3, He^3	$T = \frac{1}{2}$
C^{14}, N^{14*}, O^{14}	$T = 1$

The latter triple includes two unstable isotopes, C^{14} and O^{14}, and an excited state of N^{14}, all with spin 0. The ground state of N^{14} is a $T = 0$, $J = 1$ state.

Although isospin was introduced in nuclear physics, it is applicable in elementary particle physics as well. In high energy interactions many particles, mostly unstable, are observed. One set of three pseudoscalar particles, the pions, π^+, π^0, π^-, form a $T = 1$ triplet, the lightest hadrons. $\pi^\pm$ are unstable particles living for about 26 nanoseconds, but this is long enough to make a beam. π^0 lives only 10^{-16} s, so no beam is possible, and π^0's can be detected only from their decay into two photons. This very different behavior is due to the electromagnetic interactions, which violate isospin symmetry.

Now if we consider scattering of π mesons off nucleons (*i.e.* protons or neutrons), the conservation of isospin will relate different amplitudes. The simplest application is the resonant state produced in $\pi^+ p$ scattering, called the Δ^{++}, at a mass of 1232 MeV. It must be a $T = 3/2$ state because it is $\pi^+ \otimes p$ or $|1, 1\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle = |3/2, 3/2\rangle$. This must be part of an irreducible representation of the (unphysical) rotations in isospace, with partners

Name	Q	rep
Δ^{++}	$2e$	$\left|\frac{3}{2}, \frac{3}{2}\right\rangle$
Δ^{+}	e	$\left|\frac{3}{2}, \frac{1}{2}\right\rangle$
Δ^{0}	0	$\left|\frac{3}{2}, -\frac{1}{2}\right\rangle$
Δ^{-}	$-e$	$\left|\frac{3}{2}, -\frac{3}{2}\right\rangle$

So observing the Δ^{++} resonance implies the existence of these three other particles, all of which are observed, though shortlived, particles.

When the Δ^{++} decays, it must always produce a p and a π^{+}, as there is no other choice. On the other hand, the decay of the Δ^{+} can give $p\pi^{0}$ or $n\pi^{+}$. The decay amplitudes are given by the Wigner-Eckhart theorem:

$$\frac{\langle \pi^0 p | \Delta^+ \rangle}{\langle \pi^+ n | \Delta^+ \rangle} = \frac{\left\langle 1, \frac{1}{2}, 0, \frac{1}{2} \big| 3/2, 1/2 \right\rangle}{\left\langle 1, \frac{1}{2}, 1, -\frac{1}{2} \big| 3/2, 1/2 \right\rangle} = \frac{\sqrt{2/3}}{\sqrt{1/3}} = \sqrt{2}.$$

The decay probability is proportional to the amplitude squared, so

$$\Delta^+ \rightarrow \pi^0 p \qquad 2/3 \text{ of the time}$$
$$\Delta^+ \rightarrow \pi^+ n \qquad 1/3 \text{ of the time.}$$

The Δ resonance is the most prominent feature of low energy πN scattering, and this prediction is the simplest. But the whole scattering amplitude can be analyzed as well. For example, the amplitude for $\pi^+ n \rightarrow \pi^0 p$, at a certain angle and with given polarization, is given by

$$\langle \pi^0 p \, \text{final} | \, S \, | \pi^+ n \, \text{init} \rangle$$

where the S or scattering matrix is isospin invariant. This amplitude will be related to others, say $\pi^+ p \rightarrow \pi^+ p$, $\pi^- p \rightarrow \pi^- p$, etc. If we write $p = N, \frac{1}{2}$ and
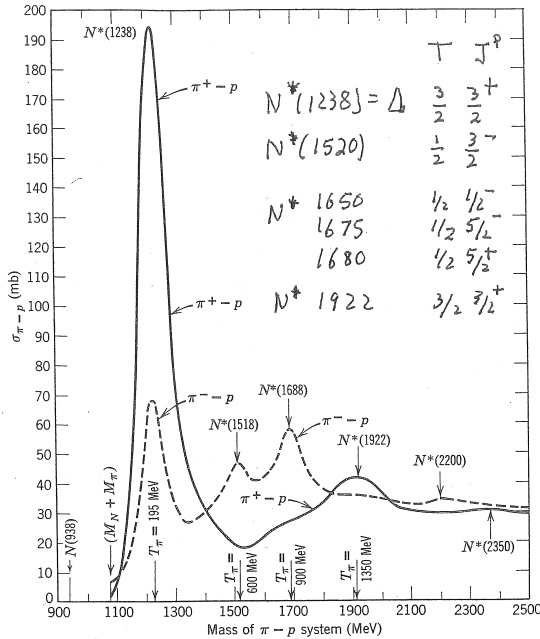
Fig. 19.3. Total cross sections for $\pi^+ p$ and $\pi^- p$ scattering.

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$n = N, -\frac{1}{2}$ we can think of the above as

$$\left\langle \pi, N, 0, \frac{1}{2} \,\Big|\, S \,\Big|\, \pi, N, 1, \frac{-1}{2} \right\rangle.$$

Inserting a complete set of isospin states, which can only have $T = \frac{1}{2}$ or $T = 3/2$, we have

$$\left\langle \pi N 0 \frac{1}{2} \,\Big\|\, \pi N \frac{3}{2}\frac{1}{2} \right\rangle \left\langle \pi N \frac{3}{2}\frac{1}{2} \,\Big|\, S \,\Big|\, \pi N \frac{3}{2}\frac{1}{2} \right\rangle \left\langle \pi N \frac{3}{2}\frac{1}{2} \,\Big\|\, \pi N 1 \frac{-1}{2} \right\rangle$$

$$+ \left\langle \pi N 0 \frac{1}{2} \,\Big\|\, \pi N \frac{1}{2}\frac{1}{2} \right\rangle \left\langle \pi N \frac{1}{2}\frac{1}{2} \,\Big|\, S \,\Big|\, \pi N \frac{1}{2}\frac{1}{2} \right\rangle \left\langle \pi N \frac{1}{2}\frac{1}{2} \,\Big\|\, \pi N 1 \frac{-1}{2} \right\rangle$$

$$= \sum_{j=\frac{1}{2},\frac{3}{2}} \left\langle 1\frac{1}{2}j\frac{1}{2} \,\Big\|\, 1\frac{1}{2}0\frac{1}{2} \right\rangle \left\langle 1\frac{1}{2}j\frac{1}{2} \,\Big\|\, 1\frac{1}{2}1\frac{-1}{2} \right\rangle a_j = \frac{\sqrt{2}}{3}a_{3/2} - \frac{\sqrt{2}}{3}a_{1/2}$$

where the $a_j = \left\langle \pi N j \frac{1}{2} \,\big|\, S \,\big|\, \pi N j \frac{1}{2} \right\rangle$ are independent of m. Is this progress? We have replaced our scattering amplitude by a combination of two undetermined parameters, but we also have the other *measureable* amplitudes

$$\pi^+ p \rightarrow \pi^+ p \;=\; a_{3/2}$$
$$\pi^- p \rightarrow \pi^- p \;=\; \frac{1}{3}a_{3/2} + \frac{2}{3}a_{1/2}$$
$$\pi^+ n \rightarrow \pi^+ n \;=\; \frac{1}{3}a_{3/2} + \frac{2}{3}a_{1/2}$$
$$\pi^- n \rightarrow \pi^- n \;=\; a_{3/2}$$
$$\pi^- p \rightarrow \pi^0 n \;=\; \frac{\sqrt{2}}{3}a_{3/2} - \frac{\sqrt{2}}{3}a_{1/2}$$

So these five measurable cross sections should all be described by two scattering amplitudes. Also determined are the undoable reactions

$$\pi^0 p \;\rightarrow\; \pi^0 p$$
$$\pi^0 p \;\rightarrow\; \pi^+ n$$
$$\pi^0 n \;\rightarrow\; \pi^- p$$
$$\pi^0 n \;\rightarrow\; \pi^0 n$$

so it definitely is progress, has been tested, and agrees well with experiment.

If we examine the conserved quantum numbers of the particles we have discussed so far, we see $Q = \frac{1}{2}B + T_3$, where B is the baryon number, one for nucleons and zero for pions.

Name	T_3	Q	Baryon number
n	$-1/2$	0	1
p	$1/2$	1	1
π^+	1	1	0
π^0	0	0	0
π^-	-1	-1	0

All of the particles we've considered so far are now considered to be made of two kinds of quarks, u and d, and their antiparticles $\bar{u}$ and $\bar{d}$. u and d form an isospin 1/2 multiplet and thus have $T_3 = \pm\frac{1}{2}$.

	T_3	Q	B
u	$1/2$	$2/3$	$1/3$
d	$-1/2$	$-1/3$	$1/3$

A Δ^{++} is made of three u quarks, so they have $Q = 2/3$ and $B = 1/3$. A proton is uud and a neutron udd. As the u and d quarks satisfy the $Q = B/2 + T_3$ relation, so will anything made up of them, as Q, T_3 and B are all arithmetically additive quantum numbers.

But not everything is made of u and d. Physics learned about strangeness in 1953 and proposed the strange quark at the same time as u and d. The strange quark has no isospin, $Q = -1/3$ and $B = 1/3$. The isospin symmetry which rotated protons into neutrons, rotating u's into d's, can be extended to a bigger approximate symmetry group SU(3) by considering rotations into the strange (s) quark direction as well as the u and d directions

But before we launch into a big discussion of SU(3), we should note that the charmed quark was discovered in 1974 and the bottom quark in 1977. Finally the sixth quark, the top, was discovered at Fermilab in 1994. So it is time to consided groups more generally, so we can do all possibilities at once.