Chapter 9

Finding the Other Roots

We have seen that that master formula is very restrictive on the possible relations between the simple roots, and gives us the restrictions on how many times a raising or lowering operator can act on a given eigenvector of the Cartan subalgebra, though only the difference of these two numbers. Recall that on a vector of weight $\vec{\mu}$, if a simple root $\vec{\alpha}^i$ can raise the weight p times and lower it q times,

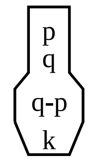
$$2\frac{\vec{\mu}\cdot\vec{\alpha}_i}{\alpha_i^2} = q - p.$$

In particular, for the adjoint representation, where the weights are roots we will now call $\vec{\phi}_k$, and where every positive root is a sum $\vec{\phi}_k = \sum_j k_j \vec{\alpha}^j$, these are determined by the **Cartan matrix** for the simple roots

$$A_{ji} = \frac{2\vec{\alpha}_j\cdot\vec{\alpha}_i}{\alpha_i^2} = q - p.$$

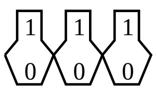
Because we know that no simple root can lower another, so the relevant $q = 0$, when one acts on another, we can determine recursively the effective raising that can be done on each root. As no two roots can have the same root vector, and as the root vectors can be found by raising with the simple roots, we can generate the whole algebra.

Georgi describes this procedure on pages 115-121, but I found his explanation a bit opaque, so I am replacing his boxes with bottles. Each box for Georgi, which represents one positive root of the algebra, contains the $q - p$ value for each simple root acting on the positive root. I will use a more complex figure, a composite of bottles for each simple root, Each bottle contains, in its widest part, the same $q^i - p^i$ Georgi puts in his box, but also the k_i at the bottom and the p^i above the q^i in the neck. Of course this is redundant information, but it helps in tracing the diagram. Each bottle contains four numbers:



1) the k_j for this root.
2) the $q - p$ for α^j on this root
3) the number of lowerings possible q
4) the number of raisings possible p

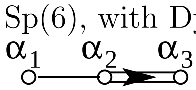
In the diagram, each positive root will consist of a string of touching bottles, one for each simple root. The diagram starts with a different symbol which represents the entire Cartan subalgebra, but with only one set of m



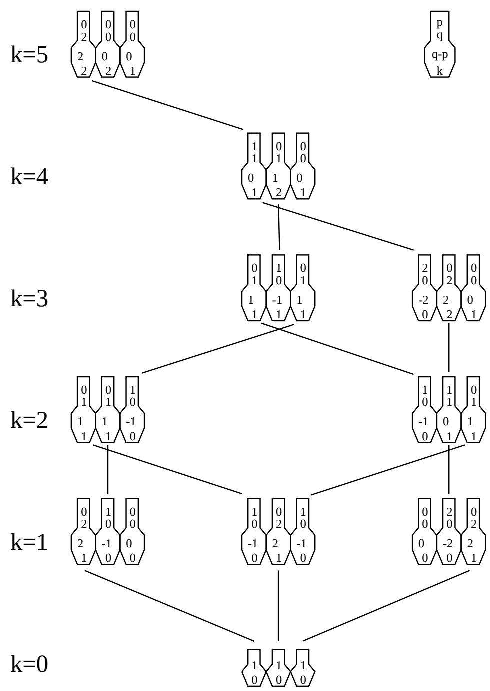
, jugs, with just p and k, where all the $k_i = 0$ and all the p^i are 1, because each simple root can act only once ($E_{2\vec{\alpha}}$ is not a root). There will be m of these truncated bottles for a rank m algebra.

The simple roots correspond to a $k = \sum k_i = 1$, one level up from the figure representing the Cartan subalgebra. For each of these roots, E_{α^i}, we have $q_j = -2\delta_{ij}$ and $p_i = 0$, because $E_{-\alpha}$, $\alpha \cdot H$, and E_α are the only states parallel to α, and no other simple root can lower a simple root, because $[E_\alpha, E_{-\beta}] = 0$ for $\alpha \neq \beta$. The $q - p$ value for the i'th bottle in the j'th simple root is A_{ji} because only $k_j \neq 0$. The p values are then determined.

Consider the example of $C_3 = \mathrm{Sp}(6)$, with Dynkin diagram



, with $(\alpha^1)^2 = (\alpha^2)^2 = 1$, $(\alpha^3)^2 = 2$, and $A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$.



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The $q - p$ values for the j'th root are just the entries in the j'th row of the Cartan matrix above. Thus we have the $k = 1$ level in the diagram shown.

When $E_{\vec{\alpha}^j}$ acts on a state μ with q^i and p^i values for $\vec{\alpha}^i$ (with $p^j \neq 0$), the state $E_{\vec{\alpha}^j} |\mu\rangle$ has $k_i \to k_i + \delta_{ij}$, $\vec{\mu} \to \vec{\mu} + \vec{\alpha}^j$ and $q^i - p^i \to q^i - p^i + A_{ji}$. So for each bottle with $p > 0$, there is a path up to one for $E_{\vec{\alpha}^j}$ acting on it with the the k_i and $q^i - p^i$ values incremented, and with its q^j one more than the q^j it came from.

A root may have more than one path leading up to it, with q^i's determined as just mentioned, and with $q^i = 0$ if there is no $E_{\vec{\alpha}^i}$ leading up to it. With all the $q^i - p^i$ and q^i so determined, the p^i are also determined, and so we can continue generating or connecting new roots until all the highest k nodes have all $p^i = 0$.

This procedure will terminate after we have generated all the positive roots. Of course the negative roots are just the negative of these, so we have generated the entire algebra.