Chapter 18

Conformal Invariance

At the beginning of the semester we motivated our investigation of symmetries by illustrating that, given differential equations which were symmetric, the solutions had to transform into each other under the symmetries as a representation of the symmetry. The first illustrations considered Schrödinger equations with symmetric potentials, such as the electrons in the spherically symmetric potential of an atom, having wavefunctions transforming under the rotation group. The Laplacian is invariant under both rotations and translations, but the source of the potential may be taken as invariant rotationally but not translationally. Under translations, we might make a connection of the wave function at $\vec{x}$ for an atom with a nucleus at $\vec{y}$ with the wave function at $\vec{x}+\vec{a}$ for an atom with a nucleus at $\vec{y}+\vec{a}$, as the physics is translationally invariant if we translate both the point of evaluation and the boundary conditions, or sources.

Thus the way symmetries act on solutions depend on both the differential operator having the symmetry and the boundary conditions or sources having the symmetry. The electric field of a point charge at the center of a conducting cube would not have a full rotational symmetry, but would behave as a representation of the symmetry group of the cube. It is obvious that the differential operator $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is symmetric under $x \leftrightarrow -x$, under $x \leftrightarrow y$, and under $x \leftrightarrow z$, which generate the group. For a spherical conductor with a charge at the center, we should have O(3) symmetry, but how do we know ∇^2 is rotationally invariant, when it seems to depend on a choice of three axes? One way is to perform a change of variables $(x, y, z) \underset{R}{\longrightarrow} (u, v, w)$ where R is an orthogonal matrix. Then the chain

rule shows that $\nabla^2 = \dfrac{\partial^2}{\partial u^2} + \dfrac{\partial^2}{\partial v^2} + \dfrac{\partial^2}{\partial w^2}$, of the same form as in terms of (x, y, z). Of course it might be even better to make a change of variables to spherical coordinates, where we would see that the r dependence factors out, and actually gives us the differential equations in θ and ϕ which would in general provide the decomposition of Φ into spherical harmonics, and for a conducting sphere would tell us Φ is independent of the angles.

The solution of a linear partial differential equation with sources in some region, with boundary conditions, for example the Poisson equation, can be found with a Green's function $G(\vec{x}, \vec{y})$ suitable to the region, $\Phi(\vec{x}) = \int d^3y G(\vec{x}, \vec{y})\rho(\vec{y})$. The invariance under symmetries is reflected in the Green's function. For example, invariance under translations tells us $G(\vec{x}, \vec{y}) = G(\vec{x}-\vec{y})$, and invariance under the full rotation group tells us that is actually a function only of $(\vec{x}-\vec{y})^2$.

To argue that ∇^2 was rotationally invariant, we showed that the form did not change under the rotation of coordinates $(x, y, z) \underset{R}{\longrightarrow} (u, v, w)$, but the form does change under $(x, y, z) \to (r, \theta, \phi)$. This change is to be considered a change of variables rather than a symmetry of the physics, and we may still write $\Phi(r, \theta, \phi) = \int \mu(r', \theta', \phi')\, G\left((r, \theta, \phi), (r', \theta', \phi')\right)\, \rho(r', \theta', \phi')$ with the correct measure $\mu = r^2 dr\, \sin(\theta)\, d\theta\, d\phi$ and a suitably modified G. This would be true, though probably not useful, even if there is no symmetry in the physics. This kind of transformation is considered a passive one, corresponding only to a change of description rather than an actual change of the physical situation under which the physics is invariant, that is, an active symmetry transformation.

This kind of coordinate independence plays a crucial role in general relativity, where we make a big point of quantities such as ∇^2 and $\nabla\Phi$ having physical meaning independent of the coordinates used to express them. This leads to the ideas of co- and contra-variant tensors, and in particular of the metric tensor, which expresses the distance ds between two positions x^μ and $x^\mu + dx^\mu$ as

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where x^μ can be any set of generalized coordinates, and ds might refer to the Minkowski rather than the Euclidean length. Under a change of coordinates, $x^\mu \to x'^\nu$, the relation of dx^μ to dx'^ν is given by the partial derivative matrix, so the metric tensor g' transforms on each index with that matrix, and $g_{\mu\nu}\, dx^\mu\, dx^\nu$ is an invariant object.

If we define $g^{\mu\nu}$ to be the inverse matrix to $g_{\mu\nu}$, *i.e.* $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$, and if we define $g := \det g_{\mu\nu}$, (or $\det(-g_{\mu\nu})$ for Minkowski space), then we can show that the invariant volume element is given by $\sqrt{g}\prod dx^\mu$ and the laplacian (or d'Alembertian) by

$$\nabla^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu} g^{\mu\nu}\sqrt{g}\frac{\partial}{\partial x^\nu}.$$

Now if we start with ordinary Euclidean or Minkowski space with cartesian coordinates, $g_{\mu\nu} = \delta_{\mu\nu}$ or $\eta_{\mu\nu}$ respectively, there are changes of coordinates which leave the **form** of $g_{\mu\nu}$ invariant, which include the translations, rotations in space, and Lorentz transformations, which suggest physics is invariant as well. The postulates of special relativity suggest that for physics to be invariant, these are the correct set of symmetry transformations. Leaving the form invariant means η is a fixed specified matrix, and the Poincaré transformations leave $(ds)^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ invariant. But the only physical law Einstein's postulate about the invariant speed of light requires is that for light in vacuum, $ds^2 = 0$. Thus a change in coordinates for which $g'_{\mu\nu}(x'^\rho) = h(x^\rho)g_{\mu\nu}(x^\rho)$ would still have that piece of physics unchanged, though different observers would not agree on the mean ds^2 lifetimes of muons.

If we consider, in flat Minkowski space, an infinitesimal transformation $x^\mu \to x'^\mu = f^\mu(x)$, and ask that two infinitesimally separated points with $ds^2 = 0$ also have $(ds')^2 = 0$, we need

$$\eta_{\mu\nu}\frac{\partial f^\mu}{\partial x^\rho}\frac{\partial f^\nu}{\partial x^\sigma}\, dx^\rho\, dx^\sigma = 0 \quad \text{whenever} \quad \eta_{\rho\sigma}dx^\rho\, dx^\sigma = 0.$$

Writing $\frac{\partial f^\mu}{\partial x^\rho} = \Lambda^\mu{}_\rho$ this is

$$\eta_{\rho\sigma}dx^\rho\, dx^\sigma = 0 \quad \text{implies} \quad \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\, dx^\rho\, dx^\sigma = 0.$$

When we were looking for Poincaré invariance, we insisted that the two sides be equal even if they were not zero, and that gave rise to the pseudo-orthogonality of Λ, but here the requirement is less restrictive and harder to interpret. But if we consider only an infinitesimal transformation $\Lambda^\mu{}_\rho = \delta^\mu_\rho + L_\mu{}^\rho$, the right hand side is

$$\left(\eta_{\rho\sigma} + \eta_{\mu\sigma}L_\mu{}^\rho + \eta_{\rho\nu}L_\nu{}^\sigma\right) dx^\rho\, dx^\sigma = 0 = (L_{\sigma\rho} + L_{\rho\sigma})\, dx^\rho\, dx^\sigma = 0$$

whenever $\eta_{\rho\sigma}dx^\rho\, dx^\sigma = 0$. An antisymmetric $L_{\sigma\rho}$ clearly satisfies that, and gives a Lorentz transformation, but also clearly $L_{\sigma\rho} = \eta_{\sigma\rho}$ or $L^\mu{}_\nu = \delta^\mu_\nu$ satisfies this condition as well. Choosing lightlike dx^ρ's with opposite spacelike components shows a symmetric $L_{\sigma\rho}$ must have $L_{0j} = 0$, and then we are left with $L_{00}|\vec{a}|^2 + L_{ij}a^i a^j = 0$ for any $\vec{a}$, so the only symmetric $L_{\mu\nu}$ which survives is η.

Notice that this is precisely a change in the scale of the metric, $g_{\mu\nu} \to h\, g_{\mu\nu}$. As a differential operator, this is $D = ix^\mu\frac{\partial}{\partial x^\mu}$, which commutes with the Lorentz transformations $\Lambda^\mu{}_\nu = x^\mu L_\mu{}^\nu\frac{\partial}{\partial x^\nu}$, but not with the momenta $P_\mu = i\frac{\partial}{\partial x^\mu}$, as $[D, P_\mu] = \frac{\partial}{\partial x^\mu} = iP_\mu$.

Thus we may add D to the Poincaré algebra to get an 11 dimensional Lie algebra, with Lorentz generators $M_{\mu\nu}$, translations P_μ, and dilatations D, and commutation relations

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\sigma}M_{\mu\rho} && (18.1)\\
[M_{\mu\nu}, P_\rho] &= i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu && (18.2)\\
[P_\mu, P_\nu] &= 0 && (18.3)\\
[P_\mu, D] &= iP_\mu && (18.4)\\
[M_{\mu\nu}, D] &= 0 && (18.5)
\end{aligned}$$

As differential operators these may be represented by

$$\begin{aligned}
M_{\mu\nu} &= -i\eta_{\mu\alpha}x^\alpha\frac{\partial}{\partial x^\nu} && (18.6)\\
P_\mu &= i\frac{\partial}{\partial x^\mu} && (18.7)\\
D &= ix^\mu\frac{\partial}{\partial x^\mu} && (18.8)
\end{aligned}$$

There is, however, another transformation which preserves the null directions. Consider the inversion $I : x^\mu \mapsto y^\mu = \frac{x^\mu}{x^2}$, for which the Jacobian matrix $\dfrac{\partial y^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu = \dfrac{\delta^\mu_\nu}{x^2} - 2\dfrac{x_\nu\, x^\mu}{(x^2)^2}$, so

$$\begin{aligned}
\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma &= \frac{1}{(x^2)^4}\eta_{\mu\nu}\left(\delta^\mu_\rho x^2 - 2x^\mu\, x_\rho\right)\left(\delta^\nu_\sigma x^2 - 2x^\nu\, x_\sigma\right)\\
&= \frac{1}{(x^2)^4}\left(\eta_{\rho\sigma}(x^2)^2 - 4x_\sigma x_\rho x^2 + 4x^2 x_\rho\, x_\sigma\right) = \frac{\eta_{\rho\sigma}}{(x^2)^2}
\end{aligned}$$

so $\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma dx^\rho dx^\sigma$ vanishes whenever $\eta_{\mu\nu}dx^\rho\, dx^\sigma$ does.

As D gives an infinitesimal scale change and I inverts, $IDI = -D$. Note I acts as a Lorentz scalar, so $[I, M_{\mu\nu}] = 0$. But I does not commute with P_μ, and in fact gives us something new and exciting.

Of course I is highly singular on the light cone, and is certainly not an infinitesimal transformation, but $I^2 = 1\!\!1$, so I times an infinitesimal generator times I is an infinitesimal generator. Let

$$K_\mu := IP_\mu I. \tag{18.9}$$

Then $e^{-ib^\mu K_\mu} = Ie^{-ib^\mu P_\mu}I$ maps

$$\begin{aligned} x^\mu &\to \frac{x^\mu}{x^2} \to \left(\frac{x^\mu}{x^2} + b^\mu = \frac{x^\mu + x^2 b^\mu}{x^2}\right) \to \frac{(x^\mu + x^2b^\mu)\, x^2}{(x^\mu + x^2 b^\mu)^2} \\ &= \frac{(x^\mu + x^2 b^\mu)}{1 + 2b_\nu x^\nu + x^2 b^2} \approx x^\mu + b^\nu (x^2 \delta^\mu_\nu - 2x^\mu x_\nu) \end{aligned}$$

so as a differential operator,

$$K_\nu = i(x^2\delta^\mu_\nu - 2x^\mu x_\nu)\frac{\partial}{\partial x^\mu}. \tag{18.10}$$

From the definition, we see that

$$\begin{aligned} [K_\mu, K_\nu] &= I[P_\mu, P_\nu]I = 0 && (18.11)\\ [M_{\mu\nu}, K_\rho] &= [M_{\mu\nu}, IP_\rho I] = I[M_{\mu\nu}, P_\rho]I = i\eta_{\mu\rho} I\, P_\nu\, I - i\eta_{\nu\rho} I\, P_\mu\, I \\ &= i\eta_{\mu\rho}K_\nu - i\eta_{\nu\rho}K_\mu && (18.12)\\ [K_\mu, D] &= [I\, P_\mu\, I, D] = I\, P_\mu\, I\, D - D\, I\, P_\mu\, I = -I\, P_\mu\, D\, I + I\, D\, P_\mu\, I \\ &= -I\,[P_\mu, D]\, I = -iI\, P_\mu\, I = -iK_\mu && (18.13)\\ [K_\mu, P_\nu] &= 2i\eta_{\mu\nu} D - 2iM_{\mu\nu} && (18.14) \end{aligned}$$

So we see that by adding in the *special conformal transformations* we have a 15 dimensional Lie algebra called the **conformal symmetry group**.

18.1 Maxwell's Equations

Electromagnetic fields are described by the 4-vector $A_\mu(x^\rho)$ and the field strength $F_{\mu\nu}(x^\rho)$ which is its antisymmetrized derivative. Under a scale transformation $x^\rho \to x'^\rho = \lambda x^\rho$, new fields $A'_\mu(x^\rho) = \lambda^d A_\mu(x'^\rho)$ and $F'_{\mu\nu}(x^\rho) = \lambda^{d+1}F_{\mu\nu}(x'^\rho)$ satisfy the same Maxwell's equations with the modified source term $j_\mu(x^\rho) = \lambda^{d+2} j_\mu(x'^\rho)$. Thus we may say that electromagnetism is scale invariant.

That electromagnetism should be scale invariant should be expected because the theory does not have any parameters with dimensions of length.

But a hint of a more surprising symmetry is familiar from the method of images. In electrostatics, if $\Phi(\vec{x})$ is a solution of Poisson's equation $\nabla^2\Phi(\vec{x}) = \rho(\vec{x})$, might we find a solution with $\Psi(\vec{x}) = (x^2)^p\Phi(\vec{y})$ where $\vec{y} = \frac{\vec{x}}{x^2}$?

$$\begin{aligned} \frac{\partial\psi}{\partial x_i} &= 2px_i\,(x^2)^{p-1}\Phi(\vec{y}) + (x^2)^p\frac{\partial y_j}{\partial x_i}\frac{\partial\Phi}{\partial y_j} \\ &= 2px_i(x^2)^{p-1}\Phi(\vec{y}) + (x^2)^p\left(\frac{\delta_{ij}}{x^2} - \frac{2x_ix_j}{x^4}\right)\frac{\partial\Phi}{\partial y_j} \\ \nabla^2\psi &= \left(6p(x^2)^{p-1} + 4p(p-1)x_i^2(x^2)^{p-2}\right)\Phi(\vec{y}) \\ &\quad +4px_i(x^2)^{p-1}\left(\frac{\delta_{ij}}{x^2} - \frac{2x_ix_j}{x^4}\right)\frac{\partial\Phi}{\partial y_j} \\ &\quad +(x^2)^p\left(\frac{-2x_i\delta_{ij}}{x^4} - \frac{6x_j + 2\delta_{ij}x_i}{x^4} + \frac{(2x_ix_j)(4x_i)}{x^6}\right)\frac{\partial\Phi}{\partial y_j} \\ &\quad +(x^2)^p\left(\frac{\delta_{ij}}{x^2} - \frac{2x_ix_j}{x^4}\right)\left(\frac{\delta_{ik}}{x^2} - \frac{2x_ix_k}{x^4}\right)\frac{\partial^2\Phi}{\partial y_j\,\partial y_k} \\ &= 2p(2p+1)(x^2)^{p-1}\Phi(\vec{y}) - 4p(x^2)^{p-2}x_j\frac{\partial\Phi}{\partial y_j} - 2(x^2)^{p-2}x_j\frac{\partial\Phi}{\partial y_j} \\ &\quad +(x^2)^{p-2}\delta_{jk}\frac{\partial^2\Phi}{\partial y_j\,\partial y_k} \\ &\underset{p=-1/2}{\longrightarrow} (x^2)^{-5/2}\,\nabla^2_y\,\Phi(\vec{y}). \end{aligned}$$

So we see that a solution to the Laplace equation, transformed by inversion, is also a solution, and in fact even the Poisson equation with a source ρ is a solution for a new ρ which transforms as a density ought (with fixed total charge).

If we consider an arbitrary coordinate transformation

$$r^\mu \to r'^\mu(r^\nu), \qquad g'_{\mu\nu}dr'^\mu dr'^\nu = g_{\mu\nu}dr^\mu dr^\nu$$

the metric tensor in the new coordinates is

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}.$$

If the new $g'_{\mu\nu}(r')$ has the same form, up to a symmetry of the theory, as $g_{\mu\nu}$, we can view $r \to r'$ not as a change of variables of a fixed physical system, but as a map from one system to another under a symmetry transformation.

Now in general $g_{\mu\nu}$, as a symmetric $D \times D$ matrix has $D(D+1)/2$ independent elements, and it is unlikely that an arbitrary transformation will produce such a change. But in two dimensions, if we treat $r = (x, y) \sim z = x + iy$ and $r' = (u, v) \sim w = u + iv$, something very special happens if the map $(x, y) \to (u, v)$ is a complex analytic function $z \to w$, for which the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \text{and also the inverse} \quad \frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \quad \frac{\partial y}{\partial u} = -\frac{\partial x}{\partial v}$$

hold. Then

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^{\,\mu}} \frac{\partial x^\sigma}{\partial x'^{\,\nu}} g_{\rho\sigma} = A_\mu{}^\rho A_\nu{}^\sigma g_{\rho\sigma},$$

where $A_\mu{}^\rho = \dfrac{\partial x^\rho}{\partial x'^{\,\mu}} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a = \dfrac{\partial x}{\partial u}$, $b = \dfrac{\partial x}{\partial v}$. Now if we started off in Euclidean space, with $g_{\mu\nu} = \delta_{\mu\nu}$, we have $g' = AgA^T = AA^T = (a^2 - b^2)\mathbb{I}$, and we see that we have just a dilation, so if our physics is scale invariant, an arbitrary analytic transformation is a symmetry.

Consider an electrostatics problem of finding the electric potential inside a container with specified potentials at each point of the wall. Let us consider this problem in two dimensions, which is physical if we really have a long cylinder with our 2-D space as cross section and we are looking for solutions uniform in the third dimension. Then inside the container we have Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 = \nabla^2 \Phi = \frac{1}{\sqrt{g_w}} \frac{\partial}{\partial w^\mu} g_w^{\mu\nu} \sqrt{g_w} \frac{\partial \Phi}{\partial w^\nu},$$

where the first expression assumes cartesian coordinates (x, y), but the second is good in any coordinates. If $g_w^{\mu\nu}$ is proportional to $\delta^{\mu\nu}$, it is $\delta^{\mu\nu}/\sqrt{g_w}$, as $g_w := \det g_{w\,\mu\nu}$. But then we see that, even though g_w is a function of $\vec{r}$, the term between the derivatives is not, and $\nabla^2\Phi = 0 \Leftrightarrow \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0$.

So if we solve the electrostatic problem in any region with any boundary potential specified, and if we can map this region with an analytic function into another region, we have solved the problem for this new region as well.

This has an application to fluid flow as well. Consider the motion of a fluid in two dimensions, with a velocity $\vec{V}(\vec{r})$ at each point. An important

class of flows is that of *irrotational flow*, for which $\vec{\nabla} \times \vec{V} = 0$, and so if the region is simply connected, we may define a *velocity potential* Φ for which $\vec{V} = \vec{\nabla}\Phi$. Then if the fluid is also incompressible, so that $d\rho/dt = 0$ and by continuity $\vec{\nabla} \cdot \rho\vec{V} = 0$, we have that Φ again satisfies Laplace's equation. So the same tool, finding analytic maps, is useful for fluid flow, at least for nonviscous irrotational incompressible fluids in two dimensions.

The idea of conformal equivalence has many applications, one dear to me in particular is in string theory. The states of string theory are described not as in field theory, with particle paths (Feynman diagrams) in space-times, but as mappings of the world-sheet, that is the world-surface of a one-dimensional string traveling through time, into to full (possibly 10 or 26 dimensional) full space-time. Just as the length of a path embedding in space is independent of the parameterization of that path, the area of the world sheet may be expressed as an integral over two parameters, but it is independent of coordinate transformations of those parameters, and hence conformally invariant. As a particular case, the analogue of a one-loop Feynman diagram for a closed string is a mapping of a torus into the full space-time. A torus is a two-dimensional surface so can be parameterized by two variables w_j, but, at least if embedded in 3-D Euclidean space, it is not a flat surface, so $g_w^{jk} \neq \delta_{ij}$. A two-dimensional surface may have several curvatures when considered as embedded in three dimensions, for example, on the inside of the torus, a path circling the hole is curved outside the dough of the donut but the circle of each cut you make if you slice it into three portions is curved around the dough. But these curvatures are not intrinsic to the surface. For example, a cylindrical surface is flat, in the sense that you can make it from a flat piece of paper, but a sphere or donut is not. So the intrinsic curvature of a cylinder is zero, of a sphere is positive, and of a donut is positive in some places (further from the center) and negative in others.

But as a two dimensional surface has only one curvature degree of freedom, and that can be modified by a position-dependent scale transformation, any two dimensional surface is conformally equivalent, locally, to a flat surface. Of course globally there may be problems. For the torus, we have periodic conditions in both parameters, so the conformal mapping is to a parallelogram with opposite sides identified. A rectangle will not work in general, because the naïve map from the rectangle into the torus might not preserve the angles given by the torus' metric (Note that conformal maps keep angles unchanged, being locally just a scale transformation).

So all (simple) toruses are conformally equivalent to some parallelogram, with some ratio of one side to the other and some specific angle. Considering the edges as complex numbers, this may be restated as one complex parameter τ which is the ratio of one edge to the other. But notice that changing which edge is in the denominator is equivalent to mapping $\tau \to -1/\tau$, and also, because the parallelogram represents periodic boundary conditions, we may consider this a period lattice, and note that replacing the numerator edge by the numerator edge plus the denominator edge, (or $\tau \to \tau + 1$) simply changes to an equivalent unit cell on the lattice. So our theory is invariant under the modular group (see homework 4, problem 1).

In field theory we sum over Feynman graphs integrating over positions of all vertices, using the functional integral interpretation of quantum mechanics. In string theory we should integrate over all conformally inequivalent toruses, or roughly speaking, over the two dimensional parameter τ. But we should not be integrating over the equivalent configurations described by the Modular group, so we need to factor out the modular equivalent configurations. Thus the correct integration over loop parameters for the closed string is to integrate over one "fundamental region" of the modular group.