General Relativity
As taught in 1979, 1983 by

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Contents

Note: This is being typed piecemeal in 2012 from handwritten notes in a red looseleaf marked 617 (1983) but may have originated in 1979

0.1 Introduction

I am, myself, an elementary particle physicist, and my interest in general relativity has come from the growth of a field of quantum gravity. Because the gravitational inderactions of reasonably small objects are so weak, quantum gravity is a field almost entirely divorced from contact with reality in the form of direct confrontation with experiment.

There are three areas of contact

1. In relativistic quantum mechanics, one usually formulates the physical quantities in terms of fields. A field is a physical degree of freedom, or variable, definded at each point of space and time. Classically we are used to thinking of the electromagnetic fields that way. Quantum mechanics associates particles with fields, so that E&M becomes the mechanics of photons, and gravity the mechanics of gravitons. These particles are then exchanged between other particles. The virtual particles may have any momentum and energy, and if one then sums up the contribution of all the low energy virtual particles one can reproduce Maxwell's and Einstein's laws. But the high energy contributions formally give divergent integrals, that is, they make the answers infinity times some function of the charge or the gravitational coupling constant. For Maxwell's theory, one can show that this infinity is unphyical in the sense that it arises only when writing the effect of interchanging photons in terms of the charge an electron would have had there been no photon interchange. Wehn on compares physical observables, there is no infinite constant. Now this is not true if one asks, for example, what the gravitational attraction between two electrons is. One finds, formally, that the force between two electrons nearly at rest is
$$F \sim \frac{Gm^2}{r^2}\left(1+\frac{Gm^2}{\hbar c}\times\infty\times f(r)+\mathcal{O}(Gm^2/\hbar c)^2\right)$$
so ignoring the second term is what any sensible person would do. Nonetheless the ∞ is bothersome to the pure of heart, and thus attempts at understanding why it must not really be there has led to much work on the consistancy of quantum gravity.

2. An intellectual difficulty exists in discussing the quantum mechanics of a particle near the Schwarzschild radius of a black hole. As we shall see, classically there is a distance, called the Schwarzschild radius, about any point mass, within which the gravitational field is so strong that nothing can get out. Quantum mechanics introduces in uncertainty in the position of such a particle, and therefore permits it, with some small probability, to tunnel out of the hole. Extremely interesting work of Hawking et all has created interest in this overlap region of quantum mechanics and general relativity.

3. The most recent advances in elementary particle physics have shown or at least strongly suggest that two of th fours undamental forces of the universe, the weak and the strong forces, are to be understood in terms of a gauge field theory. Another of the four forces, electromagnetism, has long been known to be a gauge theory. Furthermore, it is now fairly clear that electromagnetism and the weak interactions are really different manifestations of a unified gauge theory. Now the last of the four forces is gravity, and Einstein's theory is a sort of gauge theory, no in the same sense as the others, but partially so. This suggests that there might be a unified theory in which gravitons, photons, the intermediate vector boson which carry the weak interactions, and the gluons which carry the strong interactions, are all united into different states, related by symmetries, of the same particle. Such a theory is also pointed to by a form of theory dreamt up by particle theorists which considers particles called fermions (such as electrons and quarks) to be related by a symmetry to particles called bosons (such as photons, gravitons, *etc.*.) It is thus conceivable that all of the particles we consider to be fundamental ar but different views of the same underlying object. These theories, known as supergravity, have one amazing extra attaction. In the expression

$$F \sim \frac{Gm^2}{r^2}\left(1+\frac{Gm^2}{\hbar c}\times\infty\times f(r)+\left(Gm^2/\hbar c\right)^2 g(r)+\mathcal{O}(Gm^2/\hbar c)^3\right),$$

one finds $f(r)=g(r)=0$, eliminating at least the first two ∞'s which any other field theory of gravity with electrons gives. Needless to say, supergravity theories have been a subject of a great deal of effort. It is also how I got seriously interested in gravity.

We will not be discussing any of these topics in this course, at least not seriously. We will be developing only the classical theory and we will treat things other than gravity aas being completely different from gravity. We will not slight the geometrical interpretation of Einstein's equation. We will also discuss the observable tests, both the classical three tests (bending of light, presession of the perihelion of Mercury, and the gravitational red shift of light) and others. These tests all involve very small effects in the weak gravitational fields which we have available in our vacinity, the solar system. But there are important consequences of the theory where fields are strong. We shall find that solutions of the equations lead to fantastic predictions, namely

1. that the universe began with an explosion from an instance when its dimensions were zero.

2. that it may, depending on how much mass there is in it, collapse again to a point, burning everything in the universe to an elemental fireball.

3. that there most likely exist smaller objects, black holes, which have collapsed to a point. Anything getting sufficiently close to such an object is irrevokably drawn in to the singularity, and no message from within this radius can ever get out.

4. there may be multiple universes, connected only by such black holes, where an observer in one universe can find out about events in the other only after he has fallen into the black hole.

A very different introduction to general relativity is given in the opening chapters of each of the texts. Please read Chapter 1 on MTW, but not terribly carefully. If you find that he hasn't really defined things so you have a firm grasp on it, don't worry. This is a general flaw in the book but especially in the first chapter — we will come bakc to the material of 1.6–1.7 and make sure tho define things. Another brief introduction to history is in Weinberg Chapter 1 sections 2-3.

0.2 Special Relativity

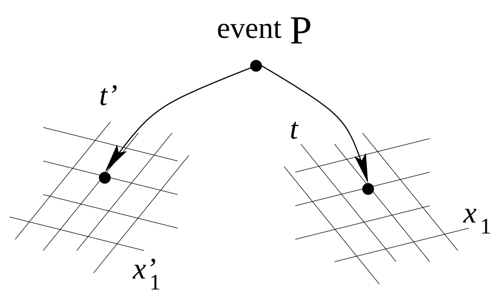
I am assuming that you have all learned special relativity in a previous course, so that this is review.

Physics transpires in spacetime. We may describe an event in spacetime by a set of coordinates x^1, x^2, x^3, t, but it is the point of spacetime and not the coordinates which has real physical meaning. A point in spacetime is called an event, whether or not anything interesting happens there.

An observer is essentially a coordinate system for describing events. Consider a particle (that is, an object of negligile spatial extension). It is associated with a locus of events, of the form "particle was at spatial point $\vec{r} = (x^1, x^2, x^3)$ at time t. [Notice the indices upstairs — this will be explained later). The locus of points is the **world line** of the particle $\vec{r}(t)$, a curve through 4 dimensional spacetime.

In special relativity we consider inertial observers in the absence of gravity. They find that free particles, which have no forces acting on them, move with constant velocity $\vec{v} = d\vec{r}/dt$. That this is possible is a law of physics, as well as a constraint on permitted coordinate systems.

Another observer, say $\mathcal{O}'$, will cover spacetime with another coordinate system. The same points, or events, in spacetime are described by $\mathcal{O}$ as (x^1, x^2, x^3, t) and by $\mathcal{O}'$ as (x'^1, x'^2, x'^3, t'), and as, at least in some region of spacetime, the coordinate quadripulates are in 1–1 correspondence with the events, we have $\vec{x}' = \vec{x}'(\vec{x}, t)$, $t' = t'(\vec{x}, t)$.



Given one inertial observer $\mathcal{O}$, and another inertial observer $\mathcal{O}'$, the requirement that one can have free particles anywhere and that both $\mathcal{O}$ and $\mathcal{O}'$ agree they are free particles means that we may write $\mathcal{O}'$'s coordinates as inhomogeneous linear functions of $\mathcal{O}$'s coordinates. Let us call $x^0 = ct$, where c is the speed of light. then let greek indices range from 0 to 3.

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}. \tag{1}$$

Note the summation convention: indices occurring once upstairs and once downstairs are implicitly summed over. If greek, $\sum_0^3$; if latin, $\sum_1^3$. Einstein said this was the "greatest contribution of my life".

The fundamental postulates which led Einstein to special relativity were

A) The laws of physics are the same in all inertial frames. All frames moving with uniform velocity (without rotation) with respect to an inertial frame are inertial.

B) The speed of light is a finite constant, c, with respect to an inertial observer.

I assume you have gone through the arguments which then lead to the Lorentz transformation. The usual sequence is

1. Lengths perpendicular to the relative motion are unchanged.
2. clocks appear to run slow by a factor of $\gamma = (1 - v^2/c^2)^{-1/2}$ for observers with respect to whom the clock is in motion with velocity v. The time interval between two events measured by an inertial clock present at both events is called the **proper time**
3. The length of measuring rods observed by someone moving with velocity v parallel to the rod is contracted by γ. the length of a rod in itw own rest frame (*i.e.* by an observer at rest with respect to the rod) is called the **proper length**.

What emerges from these considerations is that inertial reference frames are interrelated by Poincaré transformations (1) where

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}, \qquad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

η is called the Lorentz metric, and we talk about the lengths of intervals as

$$(\Delta\tau)^2 = \eta_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu},$$

despite the fact that it is not positive definite.

The Poincaré tranformations form a group[1] . That is, if $g_1 : x^{\mu} \to x'^{\mu} = \Lambda_1{}^{\mu}{}_{\nu} x^{\nu} + b_1^{\mu}$ and $g_2 : x'^{\mu} \to x''^{\mu} = \Lambda_2{}^{\mu}{}_{\nu} x'^{\nu} + b_2^{\mu}$, then

$$g_2 g_1 = g_2 \odot g_1 : x \to x''$$

[1]Define a group: a set G of elements with product rule such that
$\forall g_1 \in G, g_2 \in G, g_1 g_2 \in G$
$\exists e \in G \ni \forall g \in G, eg = g$
$\forall g_1 \in G, \exists g_2 \in G \ni g_2 g_1 = e$.

is also a Poincaré tranformation. Also every Poincaré transformation has its inverse.

The Poincaré transformations be thought of as consisting of two types. One is translations: $x'^{\mu} = x^{\mu} + a^{\mu}$, which correspond to simply moving the origin of the coordinate system (in both $\vec{x}$ and t) by $-b^{\mu}$. The second ingredient is the Lorentz transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$, which leaves the origin unchanged but "rotates" the x^{ν}. We see that

$$x'^{\mu} \eta_{\mu\nu} x'^{\nu} = \Lambda^{\mu}{}_{\rho} x^{\rho} \, \eta_{\mu\nu} \, \Lambda^{\nu}{}_{\sigma} x^{\sigma} = x^{\rho} \eta_{\rho\sigma} x^{\sigma},$$

so Λ is a transformation which preserves the lengths of x^{μ}. For intervals Δx^{μ}, the a terms cancel, so the entire Poincaré group leaves invariant the lengths of intervals

$$(\Delta\tau)^2 = \eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu}. \tag{2}$$

The Lorentz transformations themselves can be thought of three-dimensionally in terms of two types:

- rotations in 3 dimensional space, $(\vec{x}\,')^i = R^i{}_j (\vec{x})^j$, $t' = t$, and
- Boosts with a velocity $\vec{v}$. In particular for $\vec{v} \parallel x$, $\Lambda = \begin{pmatrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Example: Let $\mathcal{O}$ have a coo-coo clock. The n'th coo-cooing is an event which occurs at $\vec{x} = 0$, $t = n$ hours. $\mathcal{O}'$ sees the coo-cooing at

$$\begin{aligned} t' &= \gamma n \\ x' &= c\gamma\beta n = v\gamma n \end{aligned}$$

The successive coo-coo's occur $\gamma > 1$ hours apart and the coo-coo appears slow. It is also moving at a velocity $v\gamma/\gamma = v$ which defines v.

This time dilation of moving clocks is not to be confused with another effect, the apparent change of freuency of coo-coos due to the time it takes the light to get to the observer. Let the coo-coo wear a miners cap, pointed back at $\mathcal{O}'$. The coo=coo pops out at $t' = \gamma n$ at $x' = v\gamma n$, but $\mathcal{O}'$ does not see this popping out until the light finally reaches him, after travelling a

time $x'/c = \beta\gamma n$ back to the origin. Thus the n'th coo-coo becomes visible at time

$$t'_{\rm vis} = \gamma n + \beta\gamma n = \frac{1+\beta}{\sqrt{1-\beta^2}} n = \sqrt{\frac{1+\beta}{1-\beta}} n$$

and the frequency of coo-cooing in dimished to

$$f' = \frac{1}{\Delta t'_{\rm vis}} = \sqrt{\frac{1-\beta}{1+\beta}} f.$$

This is called the relativistic Doppler shift or red shift, because for visible light lowering the frequency means shifting the color of the light towards the red. If $\beta < 0$, we have a blue shift. These words are used to describe lowering and raising the frequency regardless of what type of frequency in involved (redshifted ultraviolet light may be made blue!)

In nonrelativistic mechanics $\vec{F} = m \frac{d^2\vec{x}}{dt^2}$. This relation should still be true in the limit that velocities are small, $\vec{v} \to 0$. We parameterize the world line of the particle $x^{\mu}(\tau)$ with the parameter $(d\tau)^2 = -\eta_{\mu\nu} dx^{\mu} dx^{\nu} = c^2 dt^2 - d\vec{x}^{\,2}$ for massive particles moving slower than the speed of light. Then in the rest frame of the particle $\tau = ct$, so we may extend the definition of the force to

$$F^{\mu} = mc^2 \frac{d^2 x^{\mu}}{d\tau^2} = mc \frac{d}{d\tau} u^{\mu} \qquad u^{\mu} = c \frac{dx^{\mu}}{d\tau}.$$

this F will therefore transform under Poincaré transformations exactly like Δx^{μ}, which makes it a "**contravariant vector**"

$$F'^{\mu} = \Lambda^{\mu}{}_{\nu} F^{\nu}.$$

[Better: The 4 velocity is defined by $m^{\mu} = c\frac{dx^{\mu}}{d\tau}$. If $v \ll c$, $\Delta\tau \approx cdt$, so $u^j = v^j$, $u^0 \approx c$. In any case, $u^{\mu}\eta_{\mu\nu}u^{\nu} = c^2 \left(-c^2 dt^2 + dx^2\right)/d\tau^2 = c^2$. So $\dfrac{d}{d\tau} m \, u^{\mu}\eta_{\mu\nu}u^{\nu} = 0 = u^{\mu}\eta_{\mu\nu}F^{\nu}$.]

Notice in nonrelativistic mechanics there is no analogue of F^0, so when $u = (1,0,0,0)$, $F = (0, \vec{F})$. that is, in the rest frame $u \cdot F u^{\mu}\eta_{\mu\nu}u^{\nu} = 0$, and the dot product of two vectors is invariant, so it must be true in all frames.

Let $P^{\alpha} = mc\frac{dx^{\alpha}}{d\tau} = mu^{\alpha}$. Then in the absence of a force, P^{α} is a constant. That makes it seem to be the momentum. In fact, for small v, $\vec{P} = m\vec{v}$, $\quad P^0 = mc + \frac{1}{2}mv^2/c = E/c$, where E includes not only the kinetic energy $\approx \frac{1}{2}mv^2$ but also the rest energy mc^2.

Notice that our m is an invariant. It is what is called the "rest mass" as opposed to the "relativistic mass", a concept which we will avoid, although it is often used in freshman courses.

Notice $u^2 = u^\mu \eta_{\mu\nu} u^\nu = \left(c\dfrac{dx^\alpha}{d\tau}\right)^2 - -c^2 \dfrac{(d\tau)^2}{(d\tau)^2} = -c^2.$

$$P^2 = P^\mu \eta_{\mu\nu} P^\nu = m^2 u^2 = -m^2 c^2 = -\frac{E^2}{c^2} + \vec{p}^{\,2}.$$

These c's are becoming very tedious, and we shall do as all realtivists do, set $c = 1$ by appropriate choice of units (measure distance in light-seconds or time in centimeters). Then $E^2 = \vec{p}^{\,2} + m^2$.

Charge currents and densities:

Consider a collection of charged point particles of charges q_n and positions $x_n^\mu(t)$. The charge density is clearly

$$\rho(\vec{x}, t) = \sum_n q_n \delta^3(\vec{x} - \vec{x}_n(t))$$

Current is a rate of flow of charge past a given plane, and can be seen to be the density times velocity for a uniformly moving body, in an argument you have probably seen several times before in E&M or thermal. Thus

$$\vec{J}(\vec{x}, t) = \sum_n q_n \delta^3(\vec{x} - \vec{x}_n(t))\vec{v}_n.$$

To make four dimensional, let $x_n^\mu(\lambda)$ be the world line in terms of an arbitray parameter λ. Define

$$J^\mu(x^\nu) = \int d\lambda \sum_n q_n \delta^4(x^\nu - x_n^\nu(\lambda)) \frac{dx^\mu}{d\lambda}.$$

If $\lambda = t$ for each world line, clearly this reduces to the previous definitions. Furthermore the definition is independent of the parameterization, for if $\tilde{x}(\tilde{\lambda}) = x(\lambda)$,

$$\int d\tilde{\lambda}\, \delta^4(x^\nu - \tilde{x}_n^\nu(\tilde{\lambda})) = \int d\lambda\, \delta^4(x^\nu - x_n^\nu(\lambda)).$$

The Dirac delta δ^4 is unchanged under $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, $x_n^\mu \to \Lambda^\mu{}_\nu x_n^\nu$, so J is a contravariant vector.

In nonrelativistic physics, we learn the conservation equation in the form $\vec{\nabla}\cdot\vec{J} + \dfrac{\partial\rho}{\partial t} = 0$, or $\displaystyle\sum_\mu \frac{\partial J^\mu}{\partial x^\mu} = 0$. To verify that,

$$\frac{\partial}{\partial x^\nu} J^\mu(x^\nu) = \int d\lambda \sum_n q_n \left[\frac{\partial}{\partial x^\nu}\delta^4(x - x_n(\lambda))\right] \frac{dx_n^\mu}{d\lambda}.$$

$$\text{Now } \frac{d}{d\lambda}\delta^4(x - x_n(\lambda)) = \frac{dx_n^\mu(\lambda)}{d\lambda}\frac{\partial}{\partial x_n^\mu}\delta^4(x - x_n(\lambda)) = -\frac{dx_n^\mu(\lambda)}{d\lambda}\frac{\partial}{\partial x^\mu}\delta^4(x - x_n(\lambda))$$

so

$$\frac{\partial}{\partial x^\mu} J^\mu = \int d\lambda \frac{d}{d\lambda}\left(\sum_n q_n \delta^4(x - x_n(\lambda))\right) = \sum_n q_n\, \delta^4(x - x_n(\lambda))\Big|_{\lambda=-\infty}^{\lambda=+\infty} = 0$$

if we assume that particles start in the infinite past and end in the infinite future, not now.

In general conserved quantities are equivalent to a divergenceless 4-current, which is therefore often called a conserved current. The total charge for such a quantity

$$Q(t) = \int d^3x\Big|_{t=\text{constant}} J^0(\vec{x}, t) \quad \text{satisfies}$$

$$\frac{dQ}{dt} = \int d^3x\, \frac{\partial}{\partial x^0} J^0 = -\int d^3x\, \vec{\nabla}\cdot\vec{J} = -\int dS\, \hat{n}\cdot\vec{J}$$

where S is a surface (at infinity) surrounding the volume over which we are calculating the charge. If it is the total charge, the volume is all of space and the surface is at infinity. We may assume, usually, that all physical events are happening with some bounded region, (at least events which affect our experiments) so we may assume $\vec{J}$ is zero as we go infinitely far away, and then $\frac{dQ}{dt} = 0$, or Q doesn't change (is conserved).

0.3 Electromagnetism

Classically and in three dimensions, electromagnetism is described by electric and magnetic fields interacting with charged particles. The laws of physics are Maxwell's equations:

$$\begin{aligned}
\vec\nabla\cdot\vec E &= \rho \\
\vec\nabla\times\vec B &= \frac{\partial\vec E}{\partial t}+\vec J \\
\vec\nabla\cdot\vec B &= 0 \\
\vec\nabla\times\vec E &= -\frac{\partial\vec B}{\partial t}
\end{aligned}$$

and the Lorentz force on a charge:

$$\vec F = \frac{d\vec p}{dt} = q\left(\vec E + \vec v\times\vec B\right).$$

Let us frist consider the latter equation acting on a particle with velocity $\vec v$ in the x direction. The rate of change of energy E is just the work done by the electric field

$$\frac{dE}{dt}=\frac{dP^0}{dt}=q\vec E\cdot\vec v = qE_x v.$$

The 4-force $f^\mu = \dfrac{dP^\mu}{d\tau}=\dfrac{dt}{d\tau}\dfrac{dP^\mu}{dt}=\gamma\dfrac{dP^\mu}{dt}$, so

$$\begin{aligned}
f^0 &= qE_x v\gamma \\
f^1 &= qE_x\gamma \\
f^2 &= qE_y\gamma - qB_z v\gamma \\
f^3 &= qE_z\gamma + qB_y v\gamma
\end{aligned}$$

Let us now view the same situation from the point of view of observer $\mathcal{O}'$ travelling with the particle. Then

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -v\gamma & 0 & 0\\ -v\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

and

$$\begin{aligned}
f'^1 &= qE_x\gamma^2 - qE_x\gamma^2v^2 = qE_x(1-v^)\gamma^2 = qE_x \\
f'^2 &= qE_y\gamma - qB_z v\gamma \\
f'^3 &= qE_z\gamma + qB_y v\gamma
\end{aligned}$$

But the Lorentz law is also valid for $\mathcal{O}'$, who sees the particle as not having any velocity, so $f'^1 = qE'_x$, $f'^2 = qE'_y$, $f'^3 = qE'_z$, so we conclude that

$$\begin{aligned}
E'_x &= E_x \\
E'_y &= E_y\gamma - B_z v\gamma \\
E'_z &= E_z\gamma + B_y v\gamma
\end{aligned}$$

We see that E does not transform like a 4-vector, and in fact that E and B are mixed up by the Lorentz transformation.

What sort of object could it be? A hint lies in thinking about the cross product $\vec v\times\vec B$. In three dimensions we may write $(\vec v\times\vec B)^i = \epsilon^{ijk}v_jB_k$, where ϵ^{ijk} is defined as the totally antisymmetric object with $\epsilon^{i23}=1$, $\epsilon^{ijk}=-\epsilon^{jik}=-\epsilon^{kji}$. But in four dimensions a cross product in impossible because the corresponding[2] ϵ has 4 indices. So it might be better to define the magnetic field as having two indices

$$B^{ij}=\epsilon^{ijk}B_k,\qquad B^{ij}=-B^{ji},$$

and write $(\vec v\times\vec B)^i = B^{ij}v_j$. B is now an antisymmetric tensor, and

$$\begin{aligned}
E'_y &= E_y\gamma - B^{xy}v\gamma \\
E'_z &= E_z\gamma + B^{xz}v\gamma.
\end{aligned}$$

Consider a tensor $F^{\mu\nu}$. It transforms the same way $A^\mu\otimes C^\nu$ does, *i.e.* $A'^\mu = \Lambda^\mu{}_\rho A^\rho$, $f'^{\mu\nu} = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma F^{\rho\sigma}$. Then

$$F'^{00} \;\sim\; A'^0\otimes C'^0 = \left(\gamma A^0 - v\gamma A^1\right)\otimes\left(\gamma C^0 - v\gamma C^1\right)$$

[2]Note added 1/31/12: The epsilon with three spatial indices transforms suitably as a tensor with three indices under rotations, and is yet unchanged. The one with four space-time indices transforms properly as a contravariant four-index tensor under Lorentz transformations, yet is unchanged. But the three index *epsilon* is not invariant under Lorentz transformations.

$$\begin{aligned}
&\sim \gamma^2F^{00}-v\gamma^2F^{01}-v\gamma^2F^{10}+v^2\gamma^2F^{11}\\
F'^{01} &\sim A'^0\otimes C'^1=\left(\gamma A^0-v\gamma A^1\right)\otimes\left(\gamma C^1-v\gamma C^0\right)\\
&\sim \gamma^2F^{01}-v\gamma^2\left(F^{00}+F^{11}\right)+v^2\gamma^2F^{10}\\
F'^{02} &\sim A'^0\otimes C'^2=\left(\gamma A^0-v\gamma A^1\right)\otimes\gamma C^2\sim\gamma F^{02}-v\gamma F^{12}\\
F'^{03} &\sim A'^0\otimes C'^3=\left(\gamma A^0-v\gamma A^1\right)\otimes\gamma C^3\sim\gamma F^{03}-v\gamma F^{13}
\end{aligned}$$

Similarly

$$\begin{aligned}
F'^{10} &= \gamma^2F^{10}-v\gamma^2\left(F^{00}+F^{11}\right)+v^2\gamma^2F^{01}\\
F'^{11} &= \gamma^2F^{11}-v\gamma^2\left(F^{01}+F^{10}\right)+v^2\gamma^2F^{00}\\
F'^{12} &= \gamma F^{12}-v\gamma F^{02}\\
F'^{ij} &= F^{ij}\quad\text{for } i=2,3,\ j=2,3
\end{aligned}$$

If the tensor is antisymmetric, this simplifies considerably:

$$\begin{aligned}
F'^{01} &= \gamma^2(1-v^2)F^{01}=F^{01}\\
F'^{02} &= \gamma F^{02}-v\gamma F^{12}\\
F'^{12} &= \gamma F^{12}-v\gamma F^{02}\\
F'^{23} &= F^{23}
\end{aligned}$$

which suggests

$$F^{01}=E_x,\quad F^{02}=E_y,\quad F^{03}=E_z$$

$$F^{12}=B^{12}=B_z,\qquad F^{13}=B^{13}=-B_y\qquad F^{23}=B^{23}=B_x$$

Let us go back to the Lorentz force of general $\vec{v}$,

$$\begin{aligned}
f^i &= \frac{d\vec{p}}{d\tau}=\gamma q\left(\vec{E}+\vec{v}\times\vec{B}\right)^i=q\left(F^{i0}\eta_{00}u^0+F^{ij}\eta_{jk}u^k\right)=qF^{i\mu}\eta_{\mu\nu}u^\nu\\
f^0 &= \frac{dP^0}{d\tau}=\gamma\frac{dE_{\text{nergy}}}{dt}=q\gamma\vec{E}\cdot\vec{v}=qF^{0i}u^i=qF^{0\mu}\eta_{\mu\nu}u^\nu,
\end{aligned}$$

so in general

$$f^\mu:=\frac{dP^\mu}{d\tau}=qF^{\mu\nu}\eta_{\nu\rho}u^\rho.$$

These η's are becoming a nuisance. We will define tensors with lower indices. First, to make what we say now about special relativity relevant later as well, call $\eta_{\mu\nu}=g_{\mu\nu}$ sometimes.

$$\begin{aligned}
A_\mu &= g_{\mu\nu}A^\nu\quad\text{for any vector}\\
F_\mu{}^\nu &= g_{\mu\rho}F^{\rho\nu}\\
F^\mu{}_\nu &= g_{\nu\rho}F^{\mu\rho}\\
F_{\mu\nu} &= g_{\mu\rho}g_{\nu\sigma}F^{\rho\sigma}
\end{aligned}$$

The notation implies the existance of a $g^{\mu\nu}$, with

$$g_{\alpha\beta}=g_{\alpha\mu}g^{\mu\nu}g_{\nu\beta}\Longrightarrow g^{\mu\nu}g_{\nu\beta}=\delta^\mu{}_\beta,\qquad \eta^{\mu\nu}=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Notice $A_\mu B^\mu=A^\mu B_\mu$, which we could write as $a\cdot B$.

We can now write $f^\mu=\dfrac{dP^\mu}{d\tau}=qF^{\mu\nu}u_\nu$.

Remember that $u\cdot f=0$? Let's check:

$$u\cdot f=q\underbrace{F^{\mu\nu}}_{\substack{\text{antisymmetric}\\\text{on interchange}\\\mu\leftrightarrow\nu}}\underbrace{u_\mu u_\nu}_{\substack{\text{symmetric}\\\text{on interchange}\\\mu\leftrightarrow\nu}}=0$$

where the (anti-) symmetry under $\mu\leftrightarrow\nu$ means it vanishes under the symmetric sum on μ and ν.

More notation: $\partial_\mu=\dfrac{\partial}{\partial x^\mu}$. The index is down because $\partial_\mu x^\nu=\delta^\nu_\mu$. Well, that's pretty nice: what about Maxwell's laws? They involve derivatives of F, so we first evaluate

$$\begin{aligned}
\partial_\mu F^{\mu0} &= -\vec{\nabla}\cdot\vec{E}=-\rho\\
\partial_\mu F^{\mu i} &= \partial_0E^i+\partial_jB^{ji}=\frac{\partial e^i}{\partial t}-\epsilon^{ijk}\partial_jB^k=-J^i\\
&\text{so}\qquad \partial_\mu F^{\mu\nu}=-J^\nu
\end{aligned}$$

constitutes two of Maxwell's equations. There remain

$$\vec{\nabla}\cdot\vec{B}=0,\qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}.$$

The first equation involves $\partial_x B_x = \partial_x B^{yz}$, so it appears to be totally antisymmetric in three indices. We have already discussed that in 4 dimensions there is no fixed totally antisymmetric tensor in 3 indices but there is one with 4,

$$\epsilon_{o123} = 1, \qquad \epsilon_{\mu\nu\rho\sigma} = -\epsilon_{\nu\mu\rho\sigma} = -\epsilon_{\rho\nu\mu\sigma} = -\epsilon_{\sigma\nu\rho\mu}.$$

[Note: don't we want the opposite choice of sign? Maybe not, this agrees with MTW (3.50e)]

We can then form the object

$$Z_\mu = \epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} \qquad \text{where} \quad \partial^\nu = g^{\nu\beta}\partial_\beta = \eta^{\nu\beta}\frac{\partial}{\partial x^\beta}$$

The zeroth component is $\epsilon_{0ijk}\partial_i F^{jk} = \epsilon_{ijk}\partial_i B^{jk}$. Recall $B^{jk} = \epsilon^{\ell jk} B_\ell$ so

$$\epsilon_{ijk}B^{jk} = \underbrace{\epsilon^{\ell jk}\epsilon_{ijk}}_{2\delta_i^\ell} B_\ell = 2B_i, \qquad \text{so} \quad Z_0 = 2\partial_i B_i = 2\vec\nabla\cdot\vec B = 0.$$

The spatial components are

$$\begin{aligned} Z_i &= \epsilon_{i\mu\rho\sigma}\partial^\mu F^{\rho\sigma} = \epsilon_{i0jk}\partial^0 F^{jk} + 2\epsilon_{ij0k}\partial_j F^{0k} \\ &= -\underbrace{\partial^0}_{-\partial_0}\underbrace{\epsilon_{ijk}B^{jk}}_{2B_i} + 2\epsilon_{ijk}\partial_j E_k = 2\left(\frac{\partial\vec B}{\partial t} + \vec\nabla\times\vec E\right)_i = 0. \end{aligned}$$

Thus the last two of Maxwell's equations are

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma}.$$

This is sometimes written in an equivalent way:

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0.$$

Summary:

$$\begin{aligned} & F^{\mu\nu} = -F^{\nu\mu}, \qquad F^{0i} = E^i, \qquad F^{ij} = \epsilon^{ijk}B_k, \\ \text{Lorentz force:} \quad & f^\mu = \frac{dP^\mu}{d\tau} = qF^{\mu\nu}u_\nu, \\ \text{Maxwell:} \quad & \epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0 \\ & \partial_\mu F^{\mu\nu} = -J^\nu \end{aligned}$$

We will return to these equations when we learn to use differential forms.

0.4 Stress-Energy Tensor

Recall that if a "charge" q_n is associated with each particle n, we may define a current

$$J^\nu(x) = \int d\lambda \sum_n q_n \delta^4\left(x - x_n(\lambda)\right)\frac{dx_n^\nu(\lambda)}{d\lambda}.$$

such currents may be written for any property of the particles, not just the electric charge. In particular, each particle has momentum p^μ, so we may write

$$T^{\mu\nu}(x) = \int d\lambda \sum_n p_n^\mu \delta^4\left(x - x_n(\lambda)\right)\frac{dx_n^\nu(\lambda)}{d\lambda}.$$

This object is called the stress-energy tensor. It is independent of the choice of parameter λ. Two special choices are

1. $\lambda = t$, $T^{\mu\nu}(\vec x, t) = \sum_n p_n^\mu \delta^3\left(\vec x - \vec x_n(t)\right)\frac{dx^\nu}{dt}$ as $\int dt'\delta(t-t') = 1$. Thus $T^{\mu j}$ is the flux of momentum p^μ across a surface perpendicular to the j direction, just as $\vec J$ is the curent per unit area across a boundary. The components $T^{\mu 0}$ are the density of the μ component of momentum.

2. $\lambda = \tau$, $T^{\mu\nu}(x) = \int d\tau \sum_n \delta^4(x - x_n(\tau)) m_n \frac{dx_n^\mu}{d\tau}\frac{dx_n^\nu}{d\tau}$. In this form we see that $T^{\mu\nu}$ is symmetric under $\mu \leftrightarrow \nu$. We also see that it is a tensor, transforming like $dx^\mu \otimes dx^\nu$.

Conservation:

$$\begin{aligned} \partial_\nu T^{\mu\nu}(x) &= \sum_n \int d\lambda\, p_n^\mu(\lambda)\frac{dx_n^\nu}{d\lambda}\underbrace{\underbrace{\frac{\partial}{\partial x^\nu}\delta^4\left(x - x_n(\lambda)\right)}_{-\frac{\partial}{\partial x_n^\nu}\delta^4\left(x - x_n(\lambda)\right)}}_{-\frac{d}{d\lambda}\delta^4\left(x - x_n(\lambda)\right)} \\ &= -\sum_n p_n^\mu(\lambda)\,\delta^4\left(x - x_n(\lambda)\right)\Big|_{\lambda=-\infty}^{\lambda=+\infty} + \sum_n \int d\lambda \delta^4\left(x - x_n(\lambda)\right)\frac{dp_n^\mu}{d\lambda}. \end{aligned}$$

The first term is zero for any finite x assuming the particle go off to infinite x, at least for x^0, as $\lambda \to \pm\infty$. In the second term we can take $\lambda = t$, so it reduces to $\sum_n \delta^3\left(\vec{x}-\vec{x}_n(t)\right) \underbrace{\frac{dp^\mu_n}{dt}}_{\frac{d\tau}{dt}f^\mu_n}$. Thus

$$\partial_\nu T^{\mu\nu}(x) = G(x) = \sum_n \delta^3\left(\vec{x}-\vec{x}_n(t)\right)\frac{d\tau_n}{dt} f^\mu_n.$$

If the particles are free, $f=0$. Even if they interact at a point,

$$\partial_\nu T^{\mu\nu}(x) \simeq \sum_{x=x_n} \underbrace{\frac{d\tau_n}{dt} f^\mu_n}_{F^\mu = \frac{dP^\mu}{dt}} = \frac{d}{dt} \sum_{\substack{tinyrmparticles \\ \text{involved}}} P^\mu_n.$$

We expect the total momentum of the colliding particles to be conserved, so $\frac{d}{dt}\sum P^\mu_n = 0$, and

$$\partial_\nu T^{\mu\nu}(x) = 0.$$

When is it not zero?

1. If theres is an external field influencing p_n
2. if the particles interact at a distance.

Action at a distance would not conserve $T^{\mu\nu}$ because momentum is then transferred out of a region without any physical flow of momentum through the walls of the region. While this is allowed by Newton's laws and required by his formulation of gravity (the forces act instantanteously) this notion violates relativity. Consider two masses at rest. Move #1 up. Newton's law of gravity, or Coulomb's law, would tell you that particle #2 immediately feels a change in the direction of the force, hence carrying a signal faster than light can travel. We know that this is not true. In electromagnetism, other forces, due to the moving charges and radiating fields, cancel the effect of the change from Coulomb's law. In fact, we know it is better to think of one charge as producing the field, changes in which can propageat only at the velocity of light, and the other charge sensing the force locally through the field.

We will assume there are no actiona at a distance mechanisms in physics, and all forces apparently such are in fact conveyed by a field. We have sofar discussed the energy momentum only of the particles, no including the energy and momentum of the field.

To see how to add this in, consider electromagnetism,

$$\begin{aligned}\partial_\nu T^{\mu\nu}_{\text{particles}}(x) &= \sum_{x=x_n} \delta^3\left(\vec{x}-\vec{x}_n(t)\right)\frac{d\tau_n}{dt}\left(f^\mu_n = q_n F^{\mu\rho}(x_n)\frac{dx_{n\,\rho}}{d\tau}\right)\\ &= \sum_n q_n \delta^3\left(\vec{x}-\vec{x}_n(t)\right) F^\mu{}_\rho(x_n)\frac{dx_{n\,\rho}}{d\tau} = F^\mu{}_\rho(x)J^\rho(x).\end{aligned}$$

[Note the order of indices is important, $F^\mu{}_\rho \neq F_\rho{}^\mu$.]

What should the stress-energy tensor of teh electromagnetic field itself be? The energy density is[3]

$$T^{00} = \frac{1}{2}\left(E^2+B^2\right) = \frac{1}{2}F^{0i}F^{0i} + \frac{1}{4}F^{ij}F^{ij},$$

and the energy flux is

$$T^{0i} = S^i = \left(\vec{E}\times\vec{B}\right)^i = F^{0j}F^i{}_j.$$

This hints that T should be quadratic in F, and depend on nothing else (except, of course, the constant matrices η and ϵ. Considering Lorentz covariance and symmetries, the only possibilities are

$$T^{\mu\nu} = aF^{\mu\rho}F^\nu{}_\rho + b\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma},$$

but then $T^{00} = aF^{0i}F^{0i} + 2bF^{0i}F^{0i} - bF^{ij}F_{ij}$, so we must have $b=-1/4$, $a+2b=1/2$, so $a=1$,

$$T^{\mu\nu}_{\text{Maxwell}} = F^{\mu\rho}F^\nu{}_\rho - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}.$$

We see that

$$\begin{aligned}\partial_\nu T^{\mu\nu}_{\text{Maxwell}} &= \left(\partial_\nu F^{\mu\rho}\right)F^\nu{}_\rho - F^{\mu\rho}J_\rho - \frac{1}{2}\eta^{\mu\nu}F^{\rho\sigma}\partial_\nu F_{\rho\sigma}\\ &= -F^{\mu\rho}J_\rho + F_{\alpha\beta}\left[\partial^\alpha F^{\mu\beta} - \frac{1}{2}\partial^\mu F^{\alpha\beta}\right].\end{aligned}$$

Note that only the part of the bracket antisymmetric under $\alpha\leftrightarrow\beta$ survives contracting with $F_{\alpha\beta}$, so

$$[] \to \frac{1}{2}\partial^\alpha F^{\beta\mu} - \frac{1}{2}\partial^\beta F^{\mu\alpha} - \frac{1}{2}\partial^\mu F^{\alpha\beta} = \frac{1}{2}\left\{\partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} + \partial^\mu F^{\alpha\beta}\right\} = 0$$

[3] We are using units with $\mu_0 = \epsilon_0 = 1$.

by the first Maxwell equation, $\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0$.
Therefore

$$\partial_\nu T^{\mu\nu}_{\text{Maxwell}} = -F^{\mu\rho} J_\rho, \quad \text{and, if} \quad T^{\mu\nu} = T^{\mu\nu}_{\text{particles}} + T^{\mu\nu}_{\text{Maxwell}}, \qquad \partial_\nu T^{\mu\nu} = 0 \quad !$$

Another property carried by a particle is its angular momentum about a given poin. Ignoring any contributions from intrinsic spin, $\vec{L} = \vec{x}_n \times \vec{p}_n$. The 3-current of such an object might then be expected to be

$$\mathcal{M}^{ijk}(x) = \sum_n \left(x^i_n p^j_n - x^j_n p^i_n \right) \delta^3 \left(x - x_n \right) \frac{dx^k_n}{dt} = x^i T^{jk}(x) - x^j T^{ik}(x).$$

To make 4-dimensional we simply define

$$\mathcal{M}^{\mu\nu\rho}(x) = x^\mu T^{\nu\rho}(x) - x^\nu T^{\mu\rho}(x),$$

and

$$\partial_\rho \mathcal{M}^{\mu\nu\rho} = \delta^\mu_\rho T^{\nu\rho} + x^\mu \underbrace{\partial_\rho T^{\nu\rho}}_{0} - \delta^\nu_\rho T^{\mu\rho} - x^\nu \underbrace{\partial_\rho T^{\mu\rho}}_{0} = T^{\nu\mu} - T^{\mu\nu} = 0$$

as T is symmetric. Thus $\mathcal{M}^{\mu\nu\rho}$ corresponds to a conserved quantity, assuming T falls off sufficiently fast at ∞. We have already implied that

$$J^{ij}(t) = \int d^3x \mathcal{M}^{ij0} = \text{angular momentum.}$$

We also have

$$J^{0k}(t) = \int d^3x \left(t T^{k0} - x^k T^{00} \right) = t p^k - \int x^k T^{00} d^3x.$$

Note the energy-weighted center of mass: $\bar{x}^k = \dfrac{\int x^k T^{00} d^3x}{E}$, so

$$J^{ok} = t p^k - \bar{x}^k E = e \left(\bar{x}^k - v^k t \right),$$

where $v^k = p^k / E$. Thus the conservation of J^{0k}, along with $\vec{p}$ and E, indicates that

$$\bar{x}^k(t) = \text{const} + v^k t,$$

or the center of energy moves with a velocity given by the usual formula in terms of the total momentum and energy.

$\mathcal{M}$ is not truly a tensor because it varies under translations, as does J. A translation-invariant object may be formed from J, $M^2 = -p^\alpha p_\alpha$:

$$W_\alpha := M S_\alpha := \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} J^{\beta\gamma} p^\delta,$$

which is the spin. As J and p are conserved if there are no external forces, so are M and S_α. M is the total mass of the system, which we can see, is just the integrated energy density in the inertial coordinate system in which $\vec{p} = 0$. S is the spin. It is invariant under a translation $x \to x + a$,

$$J^{\mu\nu} = \sin(x+a)^\mu T^{\nu 0} - (x+a)^\nu T^{\mu 0} = J^{mu\nu} + a^\mu p^\nu - a^\nu p^\mu,$$

$$M S_\alpha \to \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \left(J^{\beta\gamma} p^\delta + a\beta p^\gamma + a\gamma p^\beta \right) p^\delta = M S_\alpha$$

because $\epsilon_{\alpha\beta\gamma\delta} p^\gamma p^\delta = 0$.

Thus S transforms like a vector. It corresponds to the spin of the system, that is, the angular momentum in the rest frame. We would expect it to have only three components, and indeed it satisfies te constraint $p^\alpha S_\alpha = \frac{1}{2} M^{-1} \epsilon_{\alpha\beta\gamma\delta} J^{\beta\gamma} p^\alpha p^\delta = 0$.

Any isolated system has a definite value of the two scalar quantities M^2 and W^2 (and, if $M^2 \neq 0, S^2 = W^2/M^2$) which are invariants under Lorentz transformations. These play a fundamental role in classifying the possible forms of quantum fields. Because spin is quantized, $S^2 = n(n+1)\hbar^2$ after quantization, and fields must transform as some representation of the Lorentz group.

We will return to the ideal gas after we discuss $T^{\mu\nu}$ as a form. I do not think we will discuss imperfect fluids and the rest of Chapter 1. (of Weinberg?)

0.5 Equivalence Principle

I am anxious to get into general relativity. We will follow the motivation of Einstein, who was clearly led to his conception of general relativity by analogy with his success in special relativity. Let us examine the beginning of his first paper on relativity:

On the Electrodynamics of Moving Bodies
by A. Einstein

> It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the con- ductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neigbbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.
>
> Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest.

Now in special relativity we restrict our attention to inertial frames. Consider the mechanics of a physicist in a closed room, which is accelerating at a constant acceleration a. From the special relativity approach we situate ourselves, $\mathcal{O}$ in an inertial frame with reapect to which he ($\mathcal{O}'$) has velocity v at a given instant. If we restrict ourselves to an interval over shich v is

small, we find every object in his room obeys $\vec{F}_i = m\dfrac{d^2\vec{x}_i}{dt^2} = m\vec{a}_i$. Using his coordinates we find $\vec{a}_i' = \vec{a}_i - \vec{a}$, so

$$m\vec{a}_i' = \text{“}\vec{F}_i'\text{”} = m\vec{a}_i - m\vec{a} = \vec{F}_i - m\vec{a}.$$

If the observer in the box tries to use Newton's laws, he looks for the physical origin of the force $\vec{F}'_i$. But the objects which are interacting with the observed object generate only the force $\vec{F}_i$, and he must postualte a pseudoforce $-m\vec{a}$ due to no definable other object. If he wishes to consclude that he must be accelerating, he must exclude the possibility that this force is due to some other object from outside. Perhaps he reasons: all other forces depend on positions, charges, and other variables of the material. But this excess force is always proportional to the mass, exactly as it would be if I were accelerating. Therefore I conclude that there are no outside influences, but I am accelerating with respect to an inertial frame.

But would he not observe exactly the same physics within his box if it was simply sitting on the surface of a large planet? Each object within the box would experience an extra force mg downwards, so that the situation wouldbe indestinguiishable from a box accelerating with $a = g$ in the opposite direction.

Now you should argue that the way real forces are distinguished from pseudoforces is that they depend on some property of the object, such as charge, rather than being proportional to the inertial mass. Perhaps the gravitational mass in $W = m_g g$ is not exactly the same as the inertial mass m_I. Any relativity book will tell you of the ingeneous experiments which attempt to find a variation in $m_g/m_I = 1 + \delta$ and show that $|\delta| < 10^{-12}$. So the masses appear to be equal. This equaltiy is so accurately known that it rules out possibilities like leaving out from m_g

- the binding energy of an atom, $\approx 10^{-8}$ in hydrogen
- the Lamb shift energy, 4×10^{-12} in hydrogen, more in otehr atoms.

So once again we have a situation with two different explanations of the same observations depending on coordinate system. Once again Einstein reaise the equivalence under certain conditions to a fundamental postulate, called the **principle of equivalence**.

Before we get too carried away, we must examin more carefully waht this equivalence is. In the box on the surface of the Earth, the objects do not really all accelerate the same, because different points are different distances away from the enter of the Earth, and the accelerations are all pointing towards the center of the Earth, and are therefore not exactly parallel to each other. If the coordinates are x^i, we will find

$$\frac{d^2x^i}{d\tau^2} = a^i(x^j) = a^i(0) + x^j\,\partial_j a^i\Big|_0 + \dots$$

where 0 is within the box, and we will think of the box's extent (range of $\vec{x}$) as small compared to the variation scale of a (that is, $a/\partial_j a$). The $a^i(0)$ term is the same for all particles in the box, and can be considered a pseudoforce due to accelteration of the box. But the second term, which gives the variation of the accelerations, is a detectale effect, driving objects towards the floor and foof and in from the sides of a satellite in free fall. These are called **tidal forces**. So we cannot say that all gravitational forces are pseudoforces, but only that the gravitational force at any particular point may be considered a pseudoforce.

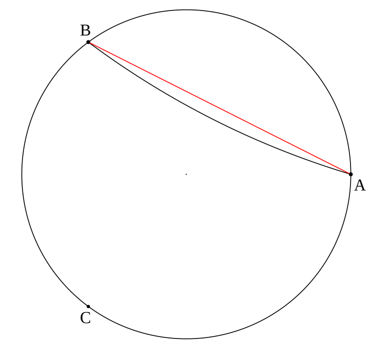
In the absence of gravity, the equations of motion are fiven by the laws of special relativity, together with whatever the relevant mechanics of the matter is. By the equivalence principle, if we can set up a coordinate system in which there are no gravitational forces, then physics obeys special relativistic laws in that coordinate system. In other coordinate systems, we must expect physics to be wierd.

We all knw that if you try to describe mechanics from an accelerating frame there are strange forces. For example, in a rotating system there are centrifugal and Coriolus forces. But there is worse.

Consider[4] a rotating table, and let observers moving with the table at-

[4] Reference: Feynman, Lectures in Physics II, chapter 42.

tempt to draw a triangle. They draw straight lines from A to B, *etc.*. What does straight line mean? The shortest distance between two points. So they draw tow paths as shown. The red line looks straight to us, but when they go to compare the lengths, they find it is longer. Why? According to us, not rotating, their metersticks shrink increasingly as they go away from the center, especially when held tangentially, so they are measuring the red line with shrunken metersticks, and more of them fit along that line than along the one that appears curved to us. If they do the same between B and C, and between C and A, and measure the angles, they will find the sum of the angles of their triangle is less than 180°! Geometry is not Euclidian or Lorentz when observed in an accelerating coordinate system.



Let us return to our box which may be accelerating through empty space or may be sitting on the surface of a large planet, with no way for us to tell which. A photon comes through a one-way window anc crosses the box. If we are an accelerating spaceship, in an inertial observer looking in sees the photon moving in a straight line, as would any other free particle, while our box accelerates upwards with acceleration g. Therefore to an observer with coordinates fixed in teh box, the photon falls with the same accelertation g as all other particles. This requires that light is bent in a gravitational field, so that, for example, star light passign the sun should be bent inwards, and stars observed on opposite sides of the sun during a solar eclipse appeart to be further appart than usual. We will return to this later, as if we did the calculation no we would get the wrong answer by a factor of two.

Another conclusion we may reach is even more startling, though not quite so simple. Suppose we have two clocks, one at the top of the spaceship-box and one at the bottom, a distance h apart. Let us observe with an inertial observer $\mathcal{O}$, at a time when the velocity of the ship is small. If the bottom clock emits a flash of light when $v = 0$, it will not be received by the top clock until time $h/c = h$, at which time the clock will be moving away from the source at $v = ah$. The light will therefore be red-shifted by

$$f_{\rm top} = f_{\rm bottom}\sqrt{\frac{1-ah}{1+ah}} \approx f_{\rm bottom}\,(1-ah)\,.$$

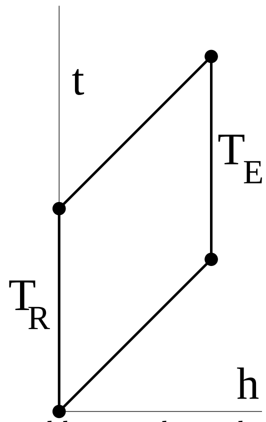
Similarly if the clock on top emits a flash when $v = 0$, the bottom one will receive it at time h, at which time it is moving twoards the source at velocity ah, and the light is blue-shifted

$$f_{\rm bottom} = f_{\rm top}\sqrt{\frac{1+ah}{1-ah}} \approx f_{\rm top}\,(1+ah)\,.$$

This agrees with the previous equation, and both observrs wagree that the frequency of ticks of the bottom clock is lower than that of the top, or the higher clock is running faster!

Now suppose our box is not a spaceship but the Empire State Building. Einstein says physics is the same, and the executives at the top are ageing faster than the receptionist on the first floor, at a rate $1 + gh = 1 + gh/c^2$ faster, which makes them about 1 μs older for each year they worked.

Although this effect is probably not the correct explanation of their gray hair, it does lead us to an interesting conclusion: spacetime as measured on a planet's surface is not Minkowskian!. If the receptionist emits light rays one second apart, each travels up the Minkowski diagram at 45°, forming a parallelogram, but $T_E > T_R$.



This was presaged by our discussion of the turntable: accelerated observers do not see Minkowskian geometry. Any hope for Minkowskian geometry can only be for an inertial observer who feels no gravity. Given any particular evernt we can alsways find such an observer by letting hime free-fall, but in his coordinate system gvarity vanishes only in the neighborhood of the chosen event. There is no way to set up a global coordinate system which in inertial, so there is no way to treat the global geometry as Minkowskian. We are goint to have to learn how to talk about curved spacetime.

0.6 Manifolds

We have seen that the spacetime in which physics acts is a curved space which can be considered flat (Minkowskian) in a small neighborhood at each point but cannot be considered flat globally[5]. In each region, I can find a coordinate system x^μ which is in $1-1$ correspondence witht he spacetime in that region. Such a $1-1$ map from a region of spacetimes to an open subset of $\mathbb{R}^4$ is called a chart. There is not necessarily a single chart which can cover the whole spacetime.

Example: The surface of a sphere. About any point, say Piscataway, I can erect a two-dimensional coordinate system and plot the corresponding point on my chart. Charts prepared by AAA and the like are usually called maps, but in math a map is a more general concept, so we revert to the older word **chart**. No chart can be prepared to cover the whole surface of the Earth.

Two coordinate systems set up to cover overlapping regions in spacetime produce a $1-1$ map between the two charts in $\mathbb{R}^4$. If these maps are continuous with continuous n'th derivatives it is called a $C^{(n)}$. If it is $C^{(n)}$ for every n,m it is called $C^{(\infty)}$.

A set $\mathcal{M}$, in our case spacetime, together with a set of charts, called an **atlas** is called a $C^{(n)}$ manifold it

- every point of $\mathcal{M}$ is included in an open set in some chart.
- For every pair of charts with overlapping domains, the map induced between the images of teh overlapping region is a $C^{(n)}$ map.

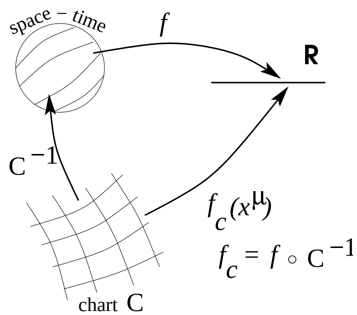
Sometimes the set $\mathcal{M}$ itslef is called the manifold, but the existance of the atlas is necessary.

Example: The surface of the world and the Rand McNally World Atlas. A simpler atlas would consist of a chart of everything above 10° S latitude together with a chart of everythin in the southern hemisphere. [I am assuming there are no overhanging cliffs in the world].

To do physics, of course, one needs more than spacetime. One also needs physical quantities defined over spacetime (fields).

[5] Refs: more formal— Chapter 2 of Hawking and Ellis, *Large Scale Structure of Space-Time.* Less formal— Misner Thorne and Wheeler, chapter 2.

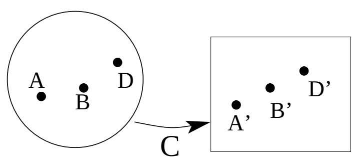
A map from spacetime into the reals, for example, the temperature in a continuous medium, induces a map from the charts (actually the range of the charts) into the reals. The physics is in the function f, but that is hard to write down. f_C is a function $\mathbb{R}^4 \to \mathbb{R}$, easy to work with, but chart dependent. A different chart will correspond to a different $f_{C'}$, even though the physics, f is the same



We now want to include vectors. We are used to thinking of a vector as an object unchanged by translations, and being a vector in a mathematical sense, that is, taking linear combinations, *etc.*.

In special relativity the difference of two points is a vector. But what does the difference of two points in a curved spacetime mean? I can subtract in $\mathbb{R}^n$, but not in $\mathcal{M}$.

Consider three points, A, B, and D, in $\mathcal{M}$, which map into three points in in $\mathbb{R}^n$. If $\Delta x = B' - A'$ and $D' - A' = 2\Delta x$, does that mean $D - A$ is twice $B - A$? Not at all, for such a statement depends on the chart C as well as any physical properties of the events A, B, and D. and $D' - A' = 2(B' - A')$ will not be true for some other chart, or choice of coordinates.



Thus we cannot define vectors as finite differences of positions on a manifold. But if we consider infinitesimal neighborhoods this problem disappears. At a given point X, we may associate a vector with a direction and a magnitude which relate to a curve passing through the point at a given rate in terms of some parameter. The way to formalize this is to treat f vector as an operator on all differentiable functions, which maps the function into its directional derivative. Thus if the curve is charted as $x^\mu(\lambda)$, the corresponding vector $\mathbf{v}(f) = \dfrac{\partial x^\mu}{\partial\lambda}\dfrac{\partial f_C}{\partial x^\mu}$, with summation over μ understood. Notice that

a) The action of $\mathbf{v}$ is independent of the chart.

b) $\mathbf{v}$ is a simple mathematical entity written without indices (though we have evaluated it with coordinates).

c) Given a particular chart C, $\partial/\partial x^\mu = \mathbf{e}_\mu$ is a set of four different vectors (not four components of one vector) which form a basis of the vector space at the point X. Any other vector $\mathbf{u} = u^\mu \mathbf{e}_\mu$, where $u^\mu = \partial x^\mu/\partial\lambda$ are th compenents of the vector $\mathbf{u}$ in the basis $\mathbf{e}_\mu$. $\mathbf{u}$ is independent of the chart chosen, but the u^μ are not.

It is important to keep in mind that in our definition, so far at least, the vector is defined at a particular point X of the manifold. Its tail is tied down, and there is no way (yet) to compare vectors defined at different points. The (possibly four-dimensional) vector space is called the tangent space at the point X.

Example: Consider a particle sensing a scalar field, for example temperature. As the particle passes the point X, at what rate does its ambient temperature change? Consider a chart. Then $x^\mu(\tau)$ is the image of its position as a function of its proper time τ, and

$$\frac{dT}{d\tau} = \frac{\partial x^\mu}{\partial\tau}\frac{\partial T_C}{\partial x^\mu} = \frac{\partial x^\mu}{\partial\tau}\mathbf{e}_\mu(T)$$

is the rate of change of temperature. This is true for any other scalar field as well, so we have an operator

$\mathbf{u}$: scalar field $\to$ proper time derivative of the field felt by a particle.

$\mathbf{u}$ is the 4-velocity of the particle! Its four components on a given chart is what we used to call a 4-vector.

Mathematicians tell us, given two vector spaces, how to define the tensor product, so there is nothing new in things like $\mathbf{u}\otimes\mathbf{v}$. Such objects are called contravariant tensors. Physically it will only be useful to consider such tensor products of vectors defined at the same point.

Another mathematical concept which is straightforward is illustrated by the metric tensor. How do we generalize the concept

$$(\Delta\tau)^2 = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 - (\Delta t)^2?$$

As we have already understood the velocity, we reconsider the equation in the form $u^\alpha u^\beta \eta_{\alpha\beta} = 1$. This must be generalized, for if η always represents the numerical matrix diag$(-1, 1, 1, 1)$, $u^\alpha u^\beta \eta_{\alpha\beta}$ has a value which depends on the chart. We need some machine which maps two vectors into the reals. Such a machine is called a covariant rank two tensor

$$g : \mathbf{u}\otimes\mathbf{v} \to \mathbb{R}.$$

Suppose we have a Minkowskian chart, that is, a coordinate system in which $u^\alpha u^\beta \eta_{\alpha\beta} = 1$. Then in this chart I can define g acting on any two vectors $\mathbf{v} = v^\mu \mathbf{e}_\mu$ and $\mathbf{w} = w^\mu \mathbf{e}_\mu$ by

$$g(\mathbf{v}, \mathbf{w}) = v^\mu w^\nu \eta_{\mu\nu}.$$

this defines a map from any two vectors in the tangent space at X into the reals. Given any other chart C', we will still find $g(\mathbf{v}, \mathbf{w})$ to be bilinear, but with changed coefficients,

$$g(\mathbf{v}, \mathbf{w}) = v'^\mu w'^\nu g'_{\mu\nu},$$

with the new metric $g'_{\mu\nu}$ in general not the fixed $\eta'_{\mu\nu}$.

If $\mathbf{u}$ is a vector defined at X, and f is a function on $\mathcal{M}$ (at least in a neighborhood of X), then $\mathbf{u}(f)$ is a real number which depends on

- $\mathbf{u}$
- the rate of change of the function f at the point X.

We may view $\mathbf{u}(f)$ as a map, determined by f and X, from the tangent space T_x of all vectors at X into the reals. Obviously not all of f is used in determining this map — it is independent of the actual value of $f(X)$ and of the values of f at other positions except insofar as they affect th rate of change at X. The map is clearly linear on T_x. Let us abstract from this the concept of a 1-form at X: a linear map from T_X into the reals.

The space of 1-forms is made into a vector space in the obvious way, $(p_1 + p_2)(\mathbf{u}) = p_1(\mathbf{u}) + p_2(\mathbf{u})$. It is not terribly big, for any 1-form $p(\mathbf{u}) = p(u^\mu(\mathbf{e}_\mu) = u^\mu p(\mathbf{e}_\mu) =: u^\mu p_\mu$, so the form is determined by the four numbers p_μ in the particular chart we used to define $(\mathbf{e}_\mu)$. The space of 1-forms at X is therefore four dimensional. To write out a basis of this space, define $\boldsymbol{\omega}^\mu$ to be a set of four distinct 1-forms with $\boldsymbol{\omega}^\mu(\mathbf{e}_\nu) = \delta^\mu_\nu$. Then $p = p_\mu \boldsymbol{\omega}^\mu$ for an arbitrary 1-form p, where p_μ are the coefficients and $\boldsymbol{\omega}^\mu$ the basis 1-forms. The basis $\{\boldsymbol{\omega}^\mu\}$ is said to be **dual** to the basis $\mathbf{e}_\mu$ of T_x.

Example: Define the 1-form $\mathbf{d}f$ by $\mathbf{d}f(\mathbf{u}) = \mathbf{u}(f) = u^\mu \partial_\mu f$. Then using $\mathbf{d}f = (df)_\mu \boldsymbol{\omega}^\mu$, $\mathbf{d}f(\mathbf{u}) = (df)_\nu \boldsymbol{\omega}^\nu(u^\mu \mathbf{e}_\mu) = (df)_\nu u^\nu$, so $(\mathbf{d}f)_\nu = \partial_\nu f$. In this chart we are therefore encouraged to call our basis 1-forms $\boldsymbol{\omega}^\mu = \mathbf{d}x^\mu$, so $\mathbf{d}f = (\partial_\mu f)\mathbf{d}x^\mu$, which is the usual expression for the differential.

We sometimes write a 1-form $\mathbf{q}$ action on a vector $\mathbf{v}$ as

$$\langle \mathbf{q}, \mathbf{v} \rangle := \mathbf{q}(\mathbf{v}).$$

Vectors have been defined in terms of linear maps from the set of functions on $\mathcal{M}$ into the reals, defined at a point P. 1-forms have been defined as maps from the space T_P of these vectors into the reals. The vectors and the forms are not dependent on the chart, but the bases we used to describe the set of these vectors and forms are. Suppose we have charts C and C' with $x^\mu = C(P)$, $x'^\mu = C'(P)$. For an arbitrary scalar function on the manifold $f(P)$, we have $f_C(x^\mu) = f_{C'}(x'^\mu)$. Let $\mathbf{u} = u^\mu \mathbf{e}_\mu = u'^\mu \mathbf{e}'_\mu$ be an arbitray vector at P. Then $\mathbf{u}(f) = u^\mu \mathbf{e}_\mu(f) = u^\mu \partial_\mu f_C|_x = u'^\mu \mathbf{e}'_\mu(f) = u'^\mu \partial'_\mu f_(C')|_{x'}$. But $\partial_\mu f_C(x) = \left.\dfrac{\partial x'^\nu}{\partial x^\mu}\right|_x \left.\dfrac{\partial f_{C'}}{\partial x'^\nu}\right|_{x'}$ by the chain rule, so

$$\mathbf{u}(f) = u^\mu \left.\frac{\partial x'^\nu}{\partial x^\mu}\right|_x \partial'_\nu f_{C'}|_{x'} = u'^\nu\, \partial'_\nu f_{C'}|_{x'}$$

from which we conclude

$$u'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} u^\mu.$$

We say that the components of a vector transform as a contravariant vector.

Example: if $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, $u'^\nu = \Lambda^\nu{}_\mu u^\mu$, so a vector has components which are also contravariant vectors in our old language.

Let $\mathbf{q}$ be a 1-form. Then

$$\begin{aligned} \mathbf{q}(\mathbf{u}) &= q_\mu \boldsymbol{\omega}^\mu(u^\nu \mathbf{e}_\nu) = q_\nu u^\nu \\ &= q'_\mu \boldsymbol{\omega}'^\mu(u'^\nu \mathbf{e}'_\nu) = q'_\nu u'^\nu = q'_\nu \frac{\partial x'^\mu}{\partial x^\nu} u^\nu. \end{aligned}$$

This must be true for any u^ν so $q_\nu = q'_\mu \dfrac{\partial x'^\mu}{\partial x^\nu}$.

Note that the chain rule guarantees $\dfrac{\partial x'^\mu}{\partial x^\nu}\dfrac{\partial x^\nu}{\partial x'^\rho} = \delta^\mu_\rho$, so we may invert this relation to get

$$q'_\mu = \frac{\partial x^\nu}{\partial x'^\rho} q_\nu.$$

Example. The inverse Lorentz transformation for x in terms of $x' x^\mu = \left(\Lambda^{-1}\right)^\mu{}_\nu (x'^\nu - a^\nu)$, so $q'_\nu = \left(\Lambda^{-1}\right)^\nu{}_\mu q_\nu$.

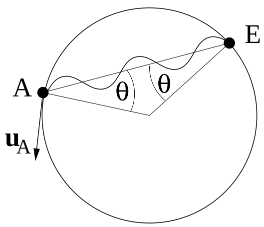
From $\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$, $\eta_{\mu\nu}\Lambda^\mu{}_\rho = \eta_{\rho\sigma}\left(\Lambda^{-1}\right)^\sigma{}_\nu$, or $\left(\Lambda^{-1}\right)^\sigma{}_\nu = \eta_{\mu\nu}\Lambda^\mu{}_\rho \eta^{\rho\sigma} =: \Lambda_\nu{}^\sigma$, $q'_\mu = \Lambda_{\mu'}{}^\nu q_\nu$, as expected for a *covariant* vector.

Thus the compenents of a 1-form transform under change of basis, as a covariant vector.

A physical example: The vave function for a particle of 4-momentum p is $f(x) \propto e^{ip_\mu x^\mu}$, whether the particle is a quantum mechanical marble, an electron, ore a photon. Let $\mathbf{u}$ be the 4-velocity of an observer. Then the vave funtion at the observer's position varies with an angular frequency $\langle p, u\rangle$ which is therefore the energy of the particle.

An example from Misner, Thorne and Wheeler: To find the red shift of a photon emitted from point E

on the rim of a turntable, and absorbed at A.



$$2\pi f_E = \mathbf{p}\cdot\mathbf{u}_E, \qquad 2\pi f_A = \mathbf{p}\cdot\mathbf{u}_A$$

In the inertial of the center, $\mathbf{p}\cdot\mathbf{u}_A = -p^0 u^0_A + |\vec{p}||vecu_A|\sin\theta$, $\mathbf{p}\cdot\mathbf{u}_E = -p^0 u^0_E + |\vec{p}||vecu_E|\sin\theta$, and $|\vec{u}_E| = \omega r\gamma = |\vec{u}_A|$, $u^0_E = u^0_A$, so $\mathbf{p}\cdot\mathbf{u}_E = \mathbf{p}\cdot\mathbf{u}_A$ and
$f_A = f_E \Longrightarrow$ there is no red shift!

A hidden assumption: The one form $\mathbf{p}$ at the event of emission E is not, *a priori*, related to the one form at A. Without defining it, we have made use of the idea that $\mathbf{p}$ is constant throughout the relevant part of spacetime.

A collection of 1-forms defined at each point in spacetime is called a **1-form field**. A collection of of tangent vectors $\in T_P$ for each P is called a **vector field**.

In special relativity we know that there is not a real diffeerence between co- and contra-variant vectors — we must simply change some signs. Thus we expect a 1-form and a vector to be similarly related. The relator is $\mathbf{g}$, the metric. Recall that $\mathbf{g}$ is a machine that maps two tangent vectors at P into the reals,

$$g : T_P \times T_P \to \mathbb{R}, \qquad \mathbf{u}\times\mathbf{v} \mapsto \mathbf{g}(\mathbf{u},\mathbf{v}).$$

Given a tangent vector $\mathbf{u}$, the map

$$U : T_P \to \mathbb{R}, \qquad \mathbf{v} \mapsto U(\mathbf{v}) := \mathbf{g}(\mathbf{u},\mathbf{v})$$

is a linear map from $T_P \to \mathbb{R}$, so it is a 1-form. If we have a Minkowskian chart, $\mathbf{g} = \eta_{\alpha\beta}\mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta$, $U(\mathbf{v}) = \eta_{\alpha\beta}u^\alpha v^\beta$, and $\mathbf{U} = \eta_{\alpha\beta}u^\alpha \mathbf{d}x^\beta$, or $U_\beta = \eta_{\alpha\beta}u^\alpha$, which is the expected relation in a Minkowski coordinate system. But

we may relate the 1-form $\mathbf{U}$ to the vector $\mathbf{u}$ in any other chart as well. There, as $\mathbf{g} = g_{\alpha\beta}\mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta$, we have

$$U_\beta = g_{\alpha\beta}u^\alpha. \qquad \text{(arbitrary frame)}$$

Recall that we first made a 1-form by considering the differential of a scalar field. Let us see what happens if we take the differential of a 1-form. Let $\mathbf{f} = f_\mu \mathbf{d}x^\mu$ be a 1-form field defined over spacetime. Let us **attempt** to define an object

$$\text{Д}\mathbf{f} = \text{Д}_{\mu\nu}\mathbf{d}x^\nu \otimes \mathbf{d}x^\mu \qquad \text{with } \text{Д}_{\mu\nu} := \frac{\partial f_\mu}{\partial x^\nu}.$$

Let us evaluate $\text{Д}\mathbf{f}$ using another chart with coordinates x'^μ. Recall $f'_\mu = \dfrac{\partial x^\nu}{\partial x'^\nu}$, so

$$\begin{aligned}
\left(\text{Д}\mathbf{f}\right)_{C'} &= \text{Д}'_{\mu\nu}\mathbf{d}x'^\nu \otimes \mathbf{d}x'^\mu \qquad \text{with} \\
\text{Д}'_{\mu\nu} &= \frac{\partial}{\partial x'^\nu} f'_\mu = \frac{\partial}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\mu} f_\rho = \frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\mu} f_\rho + \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\mu}\text{Д}_{\rho\sigma} \\
\mathbf{d}x'^\mu \otimes \mathbf{d}x'^\nu &= \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\mathbf{d}x^\rho \otimes \mathbf{d}x^\sigma, \qquad \text{so} \\
\left(\text{Д}\mathbf{f}\right)_{C'} &= \left(\text{Д}\mathbf{f}\right)_C + \frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\mu} f_\rho \mathbf{d}x'^\mu \otimes \mathbf{d}x'^\nu.
\end{aligned}$$

The last term is not zero, so we see that we have **not** obtained an object which is chart invariant — it is not physical.

We could eliminate this miserable form if we defined our rank two tensor $\text{Д}\mathbf{f}$ to be antisymmetric. We then call it $\mathbf{d}\mathbf{f}$ and write

$$\begin{aligned}
\mathbf{d}f &= (df)_{\mu\nu}\,\mathbf{d}x^\nu \otimes \mathbf{d}x^\mu, \\
\text{where} \qquad (df)_{\mu\nu} &= \frac{\partial f_\mu}{\partial x^\nu} - \frac{\partial f_\nu}{\partial x^\mu}.
\end{aligned}$$

Our chart change then gives

$$(\mathbf{d}f)_{C'} = (\mathbf{d}f)_C + \underbrace{\left(\frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\mu} - \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu}\right)}_{=0 \text{ by antisymmetry}} f_\rho \mathbf{d}x'^\mu \otimes \mathbf{d}x'^\nu,$$

so $\mathbf{d}f$ is a chart-independent real physical object.

We may also write $\mathbf{d}f = \partial_\nu f_\mu \left(\mathbf{d}x^\nu \otimes \mathbf{d}x^\mu - \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu\right)$ and introduce the **wedge product** notation

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} := \boldsymbol{\alpha} \otimes \boldsymbol{\beta} - \boldsymbol{\beta} \otimes \boldsymbol{\alpha}$$

for 1-forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, so

$$\mathbf{d}f = \partial_\nu f_\mu \mathbf{d}x^\nu \wedge \mathbf{d}x^\mu.$$

These objects are called **2-forms**.

We can keep going: An n form can be made from n 1-forms $\boldsymbol{\alpha}_i$ by taking the antisymmetric product

$$\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \ldots \wedge \boldsymbol{\alpha}_n = \sum_{\sigma \in S_n} (-1)^\sigma \boldsymbol{\alpha}_{\sigma(1)} \otimes \boldsymbol{\alpha}_{\sigma(2)} \otimes \ldots \otimes \boldsymbol{\alpha}_{\sigma(n)},$$

where S_n is the set of permutations on n elements (here $1, 2, \ldots n$), and the sign $(-1)^\sigma$ is $+1$ if σ is built of an even number of transpositions and -1 if from an odd number. .

But remember that there are really only four independent 1-forms at a point in spacetime, so antisymmetrizing in more than 4 indices gives 0.

Another description of an n form is with coefficients

$$\boldsymbol{\omega} = \frac{1}{n!} \omega_{\mu_1 \mu_2 \ldots \mu_n} \mathbf{d}x^{\mu_1} \wedge \mathbf{d}x^{\mu_2} \wedge \ldots \wedge \mathbf{d}x^{\mu_n}$$

where the coefficient $\omega_{\mu_1 \mu_2 \ldots \mu_n}$ is antisymmetric in all its indices, and the $1/n!$ cancels the fact that each $\mathbf{d}x^{\mu_1} \otimes \ldots \otimes \mathbf{d}x^{\mu_n}$ occurs $n!$ times in the sum. Thus it can't have more indices than there are dimensions. So for $\mathcal{M}$ spacetime, a 4-form is the highest we can go.

We have already defined the $\mathbf{d}$ operator on a scalar function (a 0-form) and on a 1-form $\mathcal{F} = f_\nu \mathbf{d}x^\nu$:

$$\begin{aligned} \mathbf{d}f &= (\partial_\mu f)\, \mathbf{d}x^\mu \\ \mathbf{d}\mathcal{F} &= (\partial_\mu f_\nu)\, \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu = (\mathbf{d}f_\nu) \wedge \mathbf{d}x^\nu \end{aligned}$$

More generally if $\boldsymbol{\phi} = \frac{1}{n!} \phi_{\mu_1 \cdots \mu_n} \mathbf{d}x^{\mu_1} \wedge \cdots \wedge \mathbf{d}x^{\mu_n}$,

$$\mathbf{d}\boldsymbol{\phi} = \frac{1}{n!} \left(\partial_\nu \phi_{\mu_1 \cdots \mu_n}\right) \mathbf{d}x^\nu \wedge \mathbf{d}x^{\mu_1} \wedge \cdots \wedge \mathbf{d}x^{\mu_n}.$$

An example: Suppose we have a two form $\mathbf{F} = F_{\mu\nu} \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$ with $F_{\mu\nu} = -F_{\nu\mu}$,

$$\begin{aligned} \mathbf{d}F &= \mathbf{d}\left(\frac{1}{2} F_{\mu\nu} \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu\right) = \frac{1}{2} \partial_\rho F_{\mu\nu} \mathbf{d}x^\rho \wedge \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \\ &= \frac{1}{2} \left(\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}\right) \mathbf{d}x^\rho \otimes \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu. \end{aligned}$$

What would $\mathbf{d}F = 0$ say? $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$, which is just the way we wrote two of Maxwell's equations! Thus we see that the field strengths of Maxwell is naturally implemented by a 2-form

$$\mathbf{F} = \frac{1}{2} F_{\mu\nu}\, \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu$$

which is why $F_{\mu\nu}$ is antisymmetric. And it is a *closed* two form

$$\mathbf{dF} = 0.$$

An n-form $\boldsymbol{\omega}$ is defined to be **closed** if $\mathbf{d}\omega = 0$.

Consider an n form $\boldsymbol{\omega}$ for $n = 0, 1, \ldots$:

$$\begin{aligned} \boldsymbol{\omega} &= \frac{1}{n!} \omega_{\mu_1 \mu_2 \ldots \mu_n} \mathbf{d}x^{\mu_1} \wedge \mathbf{d}x^{\mu_2} \wedge \ldots \wedge \mathbf{d}x^{\mu_n} \\ \mathbf{d}\boldsymbol{\omega} &= \frac{1}{n!} \left(\partial_{\nu_1} \omega_{\mu_1 \mu_2 \ldots \mu_n}\right) \mathbf{d}x^{\nu_1} \wedge \mathbf{d}x^{\mu_1} \wedge \mathbf{d}x^{\mu_2} \wedge \ldots \wedge \mathbf{d}x^{\mu_n} \\ \mathbf{d}\,\mathbf{d}\boldsymbol{\omega} &= \frac{1}{n!} \left(\underbrace{\partial_{\nu_2} \partial_{\nu_1}}_{\substack{\text{symmetric} \\ \nu_1 \leftrightarrow \nu_2}} \quad \omega_{\mu_1 \mu_2 \ldots \mu_n} \right) \underbrace{\mathbf{d}x^{\nu_2} \wedge \mathbf{d}x^{\nu_1}}_{\substack{\text{antisymmetric} \\ \nu_1 \leftrightarrow \nu_2}} \wedge \mathbf{d}x^{\mu_1} \wedge \mathbf{d}x^{\mu_2} \wedge \ldots \wedge \mathbf{d}x^{\mu_n} = \end{aligned}$$

so $\mathbf{d}\,\mathbf{d}\,\boldsymbol{\omega} = 0$ for any n-form $\boldsymbol{\omega}$, and $\mathbf{d}\boldsymbol{\omega}$ is always a closed $(n+1)$-form.

An n-form which can be written as $\mathbf{d}f$ for some $(n-1)$-form $\mathbf{f}$ is called **exact**. So every exact n-form is closed.

Theorem (Poincaré): Every closed n-form defined on a simply connected convex region is exact.

Consequence 1: If $\mathbf{F}$ is the electromagnetic field strength 2-form, there exists a 1-form $\mathbf{A} = A_\mu \mathbf{d}x^\mu$ such that $\mathbf{F} = \mathbf{dA}$,

$$\begin{aligned} \text{or} \quad \frac{1}{2} F_{\mu\nu} \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu &= \partial_\mu A_\nu \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \\ \text{or} \quad F_{\mu\nu} \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu &= \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu \\ \text{or} \quad F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned}$$

Thus the existence of the electromagnetic "vector potential" is a consequence of an extremely general theorem.

Consequence 2: $\mathbf{A}$ is not unique. For another $\mathbf{A}' = \mathbf{A} + \mathbf{d}\Lambda$, with Λ an arbitrary function, gives the same $\mathbf{F}$:

$$\mathbf{F}' = \mathbf{d}A' = \mathbf{d}A + \mathbf{d}\,\mathbf{d}\Lambda = \mathbf{d}A = \mathbf{F}.$$

Written in terms of components,

$$A'_\mu = A_\mu + \partial_\mu \Lambda.$$

This is known as a local gauge transformation.

A 2-form is designed to have vectors plugged in. If a particle of charge q has 4-velocity $\mathbf{u}$, what is

$$q\mathbf{F}(\,,\mathbf{u}) = qF_{\mu\nu}\left(\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu\right)(\,,\mathbf{u}) = qF_{\mu\nu}\mathbf{d}x^\mu\left(\mathbf{d}x^\nu(\mathbf{u})\right) = qF_{\mu\nu}u^\nu \mathbf{d}x^\mu = f_\mu \mathbf{d}x^\mu$$

where f_μ is the Lorentz force. Thus the 1-form $\mathbf{f} = f_\mu \mathbf{d}x^\mu = \mathbf{F}(\,,\mathbf{u})$ is the Lorentz force law.

We have now discussed, as forms or tangent vectors, all of the objects we were concerned with in special relativity, $(u^\mu, f_\mu, F_{\mu\nu}, \eta_{\mu\nu})$, except for J_μ and $\epsilon_{\mu\nu\rho\sigma}$. If we have a Minkowskian chart, we know that defining $\epsilon_{\mu\nu\rho\sigma} = \mathrm{sign}\begin{pmatrix} 0 & 1 & 2 & 3 \\ \mu & \nu & \rho & \sigma \end{pmatrix}$ is a rank four tensor under proper *Lorentz* transformations, even though its components do not change. That is to say,

$$\boldsymbol{\epsilon} := \epsilon_{\mu\nu\rho\sigma}\, \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu \otimes \mathbf{d}x^\rho \otimes \mathbf{d}x^\sigma$$

is a 4-form, and in a Minkowskian chart, $\epsilon_{0123} = 1$, but this statement is not chart independent. Recall from homework that

$$\epsilon'_{\mu\nu\rho\sigma} = \det\left(\frac{\partial x^\alpha}{\partial x'^{\,\beta}}\right)\epsilon_{\mu\nu\rho\sigma},$$

so that if x^μ are Minkowskian coordinates, $\epsilon'_{0123} = \det\left(\frac{\partial x^\alpha}{\partial x'^{\,\beta}}\right)$. This is not 1 in general, and $\epsilon'_{\mu\nu\rho\sigma}$ may no longer be considered a numerical tensor. This, of course, is also true of $g_{\mu\nu}$. For g we have

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^{\,\mu}}\,\frac{\partial x^\beta}{\partial x'^{\,\nu}}\,\eta_{\alpha\beta}$$

with x^μ Minkowskian. Notice that, considered as a matrix,

$$g' := \det\left(-g'_{\mu\nu}\right) = \left[\det\left(\frac{\partial x^\alpha}{\partial x'^{\,\beta}}\right)\right]^2 \det\left(\eta_{\alpha\beta}\right) = \left[\det\left(\frac{\partial x^\alpha}{\partial x'^{\,\beta}}\right)\right]^2,$$

so $\epsilon'_{0123} = \sqrt{g'}$ in any coordinate system.

Duality: We have already seen that the metric $\mathbf{g}$ permits making a 1-form out of a vector. We will insist always that $g_{\alpha\beta}$ be invertible ($g \neq 0$) so we can do the reverse — make a vector out of a 1-form. One need only raise the indices of the components by using $g^{\mu\nu}$ which is the inverse matrix to $g_{\mu\nu}$, *i.e.* $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$.

One can also raise the indices on an n form to make an n-vector, that is, a totally antisymmetric rank n contravariant tensor. This can then be plugged into some of the slots of the $\boldsymbol{\epsilon}$ 4-form to generate a $(4-n)$-form. This process is called **Hodge duality**, duality in a different sense than the word was used before to relate 1-forms and vectors.

It is easier to discuss in terms of components. First consider a 1-form $\mathbf{J} = J_\mu \mathbf{d}x^\mu$. Its dual is

$${}^*\mathbf{J} = \frac{1}{3!}\epsilon_{\mu\alpha\beta\gamma}J^\mu \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \wedge \mathbf{d}x^\gamma$$

which is clearly a 3-form.

From the two form $\mathbf{F} = \frac{1}{2}F_{\mu\nu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu$ we find the dual

$${}^*\mathbf{F} = \frac{1}{2!}\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}\mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta.$$

From the 3-form $\mathbf{K} = \frac{1}{3!}K_{\mu\nu\rho}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^\rho$ we find the dual

$${}^*\mathbf{K} = \frac{1}{3!1!}\epsilon_{\mu\nu\rho\alpha}K^{\mu\nu\rho}\mathbf{d}x^\alpha.$$

From the 4-form $\mathbf{G} = \frac{1}{4!}G_{\mu\nu\rho\sigma}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^\rho \wedge \mathbf{d}x^\sigma$ we find the dual

$${}^*\mathbf{G} = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}G_{\mu\nu\rho\sigma}$$

which is a 0-form or ordinary function.

Finally, from a 0-form or ordinary function f, we have

$$^*\mathbf{f} = f\boldsymbol{\epsilon} = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^\rho \wedge \mathbf{d}x^\sigma.$$

Note that $^{**}\boldsymbol{\omega} = (-1)^n\boldsymbol{\omega}$ if $\boldsymbol{\omega}$ is an n-form. [6]

If $\mathbf{F}$ is the electromagnetic field strength tensor (Faraday according to MTW) then

$$^*\mathbf{F} = \frac{1}{4}\epsilon_{\alpha\beta}{}^{\gamma\delta}F_{\gamma\delta}\mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta = \frac{1}{2}\left(^*F\right)_{\alpha\beta}\mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta,$$

$$\left(^*F\right)_{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}.$$

$^*\mathbf{F}$ is called *Maxwell* in MTW, but I don't believe this (or *Faraday* is commonly accepted notation.

What is $\mathbf{d}\,^*\mathbf{F}$?

$$\frac{1}{2}\partial_\rho\left(^*F\right)_{\mu\nu}\mathbf{d}x^\rho \wedge \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu = \frac{1}{4}\,\epsilon_{\mu\nu}{}^{\alpha\beta}\,\partial_\rho F_{\alpha\beta}\,\mathbf{d}x^\rho \wedge \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu.$$

That doesn't seem too familiar, so let us take its Hodge dual,

$$\begin{aligned} ^*\left(\mathbf{d}\,^*\mathbf{F}\right) &= \underbrace{\epsilon_{\mu\nu\rho\sigma}\frac{1}{4}\epsilon^{\mu\nu\alpha\beta}}_{-\frac{1}{2}\left(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho\right)}\,\partial^\rho F_{\alpha\beta}\,\mathbf{d}x^\sigma \\ &= -\left(\frac{1}{2}\partial^\alpha F_{\alpha\sigma} - \frac{1}{2}\partial^\beta F_{\sigma\beta}\right)\mathbf{d}x^\sigma \\ &= -\partial^\alpha F_{\alpha\sigma}\,\mathbf{d}x^\sigma = +J_\sigma\mathbf{d}x^\sigma. \end{aligned}$$

This, together with $\mathbf{d}F = 0$, are Maxwell's equations. Define $\mathbf{J} = J_\sigma\mathbf{d}x^\sigma$ to be the current density 1-form. then $^*\left(\mathbf{d}\,^*\mathbf{F}\right) = \mathbf{J}$, so $\mathbf{d}\,^*\mathbf{F} = \mathbf{d}^*\mathbf{J}$. So we have

[6] In n dimensions,

$$^*\left(\mathbf{d}x^{\mu_1} \wedge \ldots \wedge \mathbf{d}x^{\mu_p}\right) = \frac{1}{(n-p)!}\,\epsilon^{\mu_1\mu_2\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_n}\,\mathbf{d}x^{\mu_{p+1}} \wedge \ldots \wedge \mathbf{d}x^{\mu_m}.$$

Applying the dual twice to a p form, $^{**}\omega = (-1)^{p(n-p)}\omega$.

now rewritten the equations of electromagnetism as

$$\begin{aligned} \mathbf{dF} &= 0 \\ \mathbf{d}\,^*\mathbf{F} &= {}^*\mathbf{J} \\ \mathbf{f} &= \mathbf{F}(\cdot, \mathbf{u}) \end{aligned}$$

The scalar field $\phi(x)$:

$\mathbf{d}\phi = (\partial_\mu\phi)\mathbf{d}x^\mu$. To make a second derivative, we can't just take $\mathbf{d}\,\mathbf{d}\,\phi$, because that is identically zero. But we can take $\mathbf{d}\,^*\mathbf{d}\phi$, which should be a 4-form, as $^*\mathbf{d}\phi$ is a 3-form

$$\begin{aligned} ^*\mathbf{d}\phi &= \frac{1}{3!}\epsilon_{\mu\alpha\beta\gamma}\partial^\mu\phi\mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \wedge \mathbf{d}x^\gamma, \qquad \text{so} \\ \mathbf{d}\,^*\mathbf{d}\phi &= \frac{1}{3!}\epsilon_{\mu\alpha\beta\gamma}\,\partial_\nu\partial^\mu\phi\,\mathbf{d}x^\nu \wedge \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \wedge \mathbf{d}x^\gamma \end{aligned}$$

Note: doesn't this assume $\epsilon_{\mu\alpha\beta\gamma}$ is a constant?

It is easier to understand the dual of this 4-form,

$$^*d\,^*d\phi = \frac{1}{3!}\underbrace{\epsilon^{\nu\alpha\beta\gamma}\epsilon_{\mu\alpha\beta\gamma}}_{3!\delta^\nu_\mu}\partial_\nu\partial^\mu\phi = -\partial^2\phi.$$

0.7 Integration of Forms

Suppose we have a k-dimensional smooth "surface" S in $\mathcal{M}$, parameterized by coordinates $(u_1, \cdots, u_k)$. We define the integral of a k-form

$$\omega^{(k)} = \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{3}$$

over S by

$$\int_S \omega^{(k)} = \int \sum_{i_1,i_2,\ldots,i_k} \omega_{i_1\ldots i_k}(x(u)) \left(\prod_{\ell=1}^{k} \frac{\partial x_{i_\ell}}{\partial u_\ell}\right) du_1\, du_2 \cdots du_k. \tag{4}$$

We had better give some examples. For $k=1$, the "surface" is actually a path $\Gamma : u \mapsto x(u)$, and

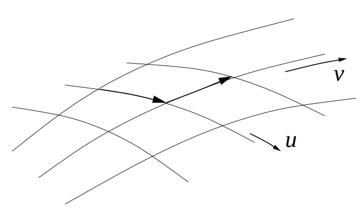
$$\int_\Gamma \sum \omega_i dx_i = \int_{u_{\min}}^{u_{\max}} \sum \omega_i(x(u)) \frac{\partial x_i}{\partial u} du,$$

which seems obvious. In vector notation this is $\int_\Gamma \vec{A} \cdot d\vec{r}$, the path integral of the vector $\vec{A}$.

For $k=2$,

$$\int_S \omega^{(2)} = \int B_{ij} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v}\, du\, dv.$$

In three dimensions, the parallelogram which is the image of the rectangle $[u, u+du] \times [v, v+dv]$ has edges $(\partial\vec{x}/\partial u)du$ and $(\partial\vec{x}/\partial v)dv$, which has an area equal to the magnitude of



$$\text{“}d\vec{S}\text{”} = \left(\frac{\partial\vec{x}}{\partial u} \times \frac{\partial\vec{x}}{\partial v}\right) du\, dv$$

and a normal in the direction of "$d\vec{S}$". Writing B_{ij} in terms of the corresponding vector $\vec{B}$, $B_{ij} = \epsilon_{ijk} B_k$, so

$$\begin{aligned}\int_S \omega^{(2)} &= \int_S \epsilon_{ijk} B_k \left(\frac{\partial\vec{x}}{\partial u}\right)_i \left(\frac{\partial\vec{x}}{\partial v}\right)_j du\, dv \\ &= \int_S B_k \left(\frac{\partial\vec{x}}{\partial u} \times \frac{\partial\vec{x}}{\partial v}\right)_k du\, dv = \int_S \vec{B} \cdot d\vec{S},\end{aligned}$$

so $\int \omega^{(2)}$ gives the flux of $\vec{B}$ through the surface.

Similarly for $k=3$ in three dimensions,

$$\sum \epsilon_{ijk} \left(\frac{\partial\vec{x}}{\partial u}\right)_i \left(\frac{\partial\vec{x}}{\partial v}\right)_j \left(\frac{\partial\vec{x}}{\partial w}\right)_k du\, dv\, dw$$

is the volume of the parallelopiped which is the image of $[u, u+du] \times [v, v+dv] \times [w, w+dw]$. As $\omega_{ijk} = \omega_{123}\, \epsilon_{ijk}$, this is exactly what appears:

$$\int \omega^{(3)} = \int \sum \epsilon_{ijk}\, \omega_{123} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v} \frac{\partial x_k}{\partial w}\, du dv dw = \int \omega_{123}(x)\, dV.$$

Notice that we have only defined the integration of k-forms over submanifolds of dimension k, not over other-dimensional submanifolds. These are the only integrals which have coordinate invariant meanings. Because the k-form is chart dependent (including that the coefficients are covariant) the expression (4) is independent of the chart C (with coordinates x^μ). But it is also independent of the parameters used to parameterize the surface. Suppose $\rho_j(\{u\})$, $j = 1, \ldots k$ is an alternate parameterization of S, with $\tilde{x}^\mu(\rho(u)) = x^\mu(u)$. Using it to define the integral,

$$\begin{aligned}\int_{S^{(\rho)}} \omega^{(k)} &= \int \sum_{i_1,i_2,\ldots,i_k} \omega_{i_1\ldots i_k}(\tilde{x}(\rho)) \left[\prod_{\ell=1}^{k} \frac{\partial x_{i_\ell}}{\partial \rho_\ell}\right] d\rho_1\, d\rho_2 \cdots d\rho_k \\ &= \int \sum_{i_1,i_2,\ldots,i_k} \omega_{i_1\ldots i_k}(\tilde{x}(\rho)) \left[\prod_{\ell=1}^{k} \left(\sum_{j_\ell} \frac{\partial x_{i_\ell}}{\partial u_{j_\ell}} \frac{\partial u_{j_\ell}}{\partial \rho_\ell}\right)\right] d\rho_1\, d\rho_2 \cdots d\rho_k \\ &= \int \sum_{i_1,i_2,\ldots,i_k} \omega_{i_1\ldots i_k}(\tilde{x}(\rho)) \left(\prod_{\ell=1}^{k} \frac{\partial x_{i_\ell}}{\partial u_\ell}\right) du_1\, du_2 \cdots du_k.\end{aligned}$$

where the antisymmetry of ω insures that the $\frac{\partial u}{\partial \rho}$'s combine, as a Jacobian, with the $\prod d\rho$ to form $\prod du$.

Thus the integral of a k-form over a k dimensional surface does not depend on how the surface is coordinatized.

Notice that we have defined this integration even on a Manifold that has no metric g. But this only permits integration of k-forms over k-dimensional surfaces, so in particular a function f can only be evaluated at points, not integrated. If we do have a metric, however, we can define on a n dimensional manifold the Levi-Civita n-form ϵ, and then integrate the Hodge dual ${}^*f$ of f over the full manifold, $\int_{\mathcal{M}} {}^*f = \int f(x^\mu) \frac{1}{n!} \epsilon_{\mu_1\mu_2\cdots\mu_n} \mathbf{d}x^{\mu_1} \wedge \mathbf{d}x^{\mu_2} \wedge \cdots \wedge \mathbf{d}x^{\mu_n}$.

We state[7] a marvelous theorem, special cases of which you have seen often before, known as **Stokes' Theorem**. Let C be a k-dimensional submanifold of $\mathcal{M}$, with ∂C its boundary. Let ω be a $(k-1)$-form. Then Stokes' theorem says

$$\int_C d\omega = \int_{\partial C} \omega. \tag{5}$$

This elegant jewel is actually familiar in several contexts in three dimensions.

If $k = 2$, C is a surface, usually called S, bounded by a closed path $\Gamma = \partial S$. If ω is a 1-form associated with $\vec{A}$, then $\int_\Gamma \omega = \int_\Gamma \vec{A} \cdot d\vec{\ell}$. Now $d\omega$ is the 2-form $\sim \vec{\nabla} \times \vec{A}$, and $\int_S d\omega = \int_S \left(\vec{\nabla} \times \vec{A}\right) \cdot d\vec{S}$, so we see that this Stokes' theorem includes the one we first learned by that name. But it also includes other possibilities. We can try $k = 3$, where $C = V$ is a volume with surface $S = \partial V$. Then if $\omega \sim \vec{B}$ is a two form, $\int_S \omega = \int_S \vec{B} \cdot d\vec{S}$, while $d\omega \sim \vec{\nabla} \cdot \vec{B}$, so $\int_V d\omega = \int \vec{\nabla} \cdot \vec{B} dV$, so here Stokes' general theorem gives Gauss's theorem. Finally, we could consider $k = 1$, $C = \Gamma$, which has a boundary ∂C consisting of two points, say A and B. Our 0-form $\omega = f$ is a function, and Stokes' theorem gives[8] $\int_\Gamma df = f(B) - f(A)$, the "fundamental theorem of calculus".

[7] For a proof and for a more precise explanation of its meaning, we refer the reader to the mathematical literature. In particular Rudin, *Principles of Mathematical Analysis* and Buck, *Advanced Calculus* are advanced calculus texts which give elementary discussions in Euclidean 3-dimensional space. A more general treatment is (possibly???) given in Spivak, *Differential Geometry*.

[8] Note that there is a direction associated with the boundary, which is induced by a direction associated with C itself. This gives an ambiguity in what we have stated, for example how the direction of an open surface induces a direction on the closed loop which bounds it. Changing this direction would clearly reverse the sign of $\int \vec{A} \cdot d\vec{\ell}$. We have not worried about this ambiguity, but we cannot avoid noticing the appearence of the sign in this last example.

0.8 Vierbeins, Connections

[Ref: Weinberg Part 2 Chapter 3]

Physics is described locally by fields, forms, and the metric tensor. At any point, the principle of equivalence tells us it is possible to choose a Minkowskian coordinate system with $g = \eta$. Let us set up a chart with coordinates ξ^α near the point $\mathcal{P}$ which is Minkowskian in the following sense:

- A free object at $\mathcal{P}$ has no acceleration in terms of the ξ coordinates, $d^2\xi^\alpha/d\tau^2 = 0$
- $\mathbf{g} = \mathbf{d}\xi^\alpha \otimes \mathbf{d}\xi^\beta \eta_{\alpha\beta}$ at $\mathcal{P}$.

[Note: the coordinates ξ^α are specially chosen to match the point $\mathcal{P}$, and more properly should be called $\xi^\alpha_\mathcal{P}$.] Einstein assures us that we can write down physics locally, at $\mathcal{P}$, in the coordinate system ξ, and it is the same as it would be were their no gravity.

The coordinates $\xi^\alpha_\mathcal{P}(\mathcal{P}')$ of the point $\mathcal{P}'$ have no decent properties except for $\mathcal{P}'$ at or near $\mathcal{P}$. In fact, we could have chosen a new chart $\xi^\alpha_{\mathcal{P}'}$ centered at $\mathcal{P}'$ to have things look Minkowskian there. Let us simultaneously use another chart $C = \{x^\mu\}$. Then

$$\mathbf{d}\xi^\alpha = V^\alpha_{\ \mu}\mathbf{d}x^\mu, \quad \text{where} \quad V^\alpha_{\ \mu} = \left.\frac{\partial \xi^\alpha}{\partial x^\mu}\right|_\mathcal{P}.$$

The object $V^\alpha_{\ \mu}(\mathcal{P})$ is called the **Vierbein**.

The components of g in C are

$$\mathbf{g} = g_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu = \eta_{\alpha\beta}\mathbf{d}\xi^\alpha \otimes \mathbf{d}\xi^\beta = \eta_{\alpha\beta}V^\alpha_{\ \mu}V^\beta_{\ \nu}\mathbf{d}\xi^\alpha \otimes \mathbf{d}\xi^\beta$$

$$\text{so} \qquad g_{\mu\nu} = \eta_{\alpha\beta}V^\alpha_{\ \mu}V^\beta_{\ \nu}.$$

The vierbein therefore determines the metric tensor.

What is the equation of motion?

$$\frac{d}{d\tau}\frac{d\xi^\alpha}{d\tau} = 0 = \frac{d}{d\tau}\left(V^\alpha_{\ \mu}\frac{dx^\mu}{d\tau}\right) = V^\alpha_{\ \mu,\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + V^\alpha_{\ \mu}\frac{d^2x^\mu}{d\tau^2} = 0$$

V is the Jacobian of a nonsingular change of variables. Its inverse is therefore

$$\left(V^{-1}\right)_\mu^{\ \alpha} = \frac{\partial x^\mu}{\partial \xi^\alpha}, \quad \text{as} \quad \left(V^{-1}\right)_\mu^{\ \alpha} V^\alpha_{\ \nu} = \frac{\partial x^\mu}{\partial \xi^\alpha}\cdot\frac{\partial \xi^\alpha}{\partial x^\nu} = \delta^\mu_\nu.$$

$$\text{Thus}\quad \frac{d^2x^\mu}{d\tau^2}+\underbrace{\left(V^{-1}\right)^\rho{}_\alpha V^\alpha{}_{\mu,\nu}}_{\Gamma^\rho{}_{\mu\nu}}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}=0$$

where we have defined the **affine connection**

$$\Gamma^\rho{}_{\mu\nu} := \left(V^{-1}\right)^\rho{}_\alpha V^\alpha{}_{\mu,\nu} = \frac{\partial x^\rho}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{6}$$

Thus we have the equation of motion

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\rho{}_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}=0 \tag{7}$$

This is also known as the geodesic equation, not only in general relativity but also on a Riemannian manifold.

Let us examine the relation of the affine connection to the metric. Note that as $\Gamma^\rho{}_{\mu\nu} := \left(V^{-1}\right)^\rho{}_\alpha V^\alpha{}_{\mu,\nu}$, $V^\alpha{}_{\mu,\nu} = V^\alpha{}_\rho \Gamma^\rho{}_{\mu\nu}$, so

$$\begin{aligned} g_{\mu\nu,\rho} &= \frac{\partial}{\partial x^\rho}\left(V^\alpha{}_\mu V^\beta{}_\nu \eta_{\alpha\beta}\right) = \left(V^\alpha{}_{\mu,\rho}V^\beta{}_\nu + V^\alpha{}_\mu V^\beta{}_{\nu,\rho}\right)\eta_{\alpha\beta} \\ &= \left(\Gamma^\sigma{}_{\mu\rho}V^\alpha{}_\sigma V^\beta{}_\nu + \Gamma^\sigma{}_{\nu\rho}V^\alpha{}_\sigma \mu V^\beta{}_\sigma\right)\eta_{\alpha\beta} = \Gamma^\sigma{}_{\mu\rho}g_{\sigma\nu} + \Gamma^\sigma{}_{\nu\rho}g_{\sigma\mu}\end{aligned}$$

Note we have assumed $\eta_{\alpha\beta,\rho} = 0$! So ξ is more than just an orthonormal set of coordinates at $\mathcal{P}$, it is also one with no acceleration without forces.

The vierbein is not a tensor, because it refers to two different charts. Γ has only indices which refer to the chart C, but nonetheless it is not a tensor. We shall see later how it changes under chart change. Nonetheless, let us raise and lower its indices with g, so

$$g_{\mu\nu,\rho} = \Gamma_{\nu\mu\rho} + \Gamma_{\mu\nu\rho}, \quad \text{but also} \quad \Gamma_{\sigma\mu\nu} = \Gamma_{\sigma\nu\mu}.$$

Add the same with $\mu \leftrightarrow \rho$ and subtract $\nu \leftrightarrow \rho$,

$$\begin{aligned} g_{\rho\nu,\mu} &= \Gamma_{\nu\rho\mu} + \Gamma_{\rho\nu\mu} = \Gamma_{\nu\mu\rho} + \Gamma_{\rho\nu\mu} \\ -g_{\mu\rho,\nu} &= -\Gamma_{\rho\mu\nu} - \Gamma_{\mu\rho\nu} = -\Gamma_{\rho\nu\mu} - \Gamma_{\mu\nu\rho}\end{aligned}$$

so, adding and dividing by two,

$$\frac{1}{2}\left(g_{\mu\nu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu}\right) = \Gamma_{\nu\mu\rho}$$

and $\Gamma^\sigma{}_{\mu\rho} = g^{\sigma\nu}\,\Gamma_{\nu\mu\rho} = \dfrac{1}{2}\, g^{\sigma\nu}\left(g_{\mu\nu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu}\right)$.

In flat space, the path of a free particle is the path which maximizes proper time (twin paradox). That means we maximize $\int_A^B d\tau$, holding the endpoints fixed. τ is an invariant, so we expect

$$S = \int_A^B d\tau = \int_A^B \sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g_{\mu\nu}(x(\lambda))}\, d\lambda$$

to be maximized along the actual classical path of the particle. We can calculate the path by varying $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$, and insist that

$$\begin{aligned} 0 = \delta S &= \int_A^B \frac{-g_{\mu\nu}\dfrac{dx^\mu}{d\lambda}\dfrac{d}{d\lambda}\delta x^\nu - \dfrac{1}{2}\dfrac{dx^\mu}{d\lambda}\dfrac{dx^\mu}{d\lambda}\dfrac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho}{\underbrace{\sqrt{}}_{} = d\tau/d\lambda}\, d\lambda \\ &= \int\left(-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d}{d\tau}\delta x^\nu - \frac{1}{2}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho\right)\frac{(d\tau/d\lambda)^2}{d\tau/d\lambda}\, d\lambda \\ &= -g_{\mu\nu}\frac{dx^\mu}{d\tau}\quad \underbrace{\delta x^\nu\big|_A^B}_{0 \text{ at } A \text{ and } B} \\ &\quad + \int\left\{g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}g_{\mu\nu,\rho}\frac{dx^\rho}{d\tau} - \frac{1}{2}\frac{dx^\mu}{d\tau}\frac{dx^\rho}{d\tau}g_{\mu\rho,\nu}\right\}\delta x^\nu\, d\tau,\end{aligned}$$

so

$$g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2} + \left(g_{\mu\nu,\rho} - \frac{1}{2}g_{\mu\rho,\nu}\right)\underbrace{\frac{dx^\mu}{d\tau}\frac{dx^\rho}{d\tau}}_{\text{symmetric in } \mu\leftrightarrow\rho} = 0$$

$$\begin{aligned} &\text{Thus}\quad \frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}g^{\mu\nu}\left(g_{\sigma\nu,\rho} + g_{\rho\nu,\sigma} - g_{\sigma\rho,\nu}\right)\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau} = 0, \\ &\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau} = 0 \qquad \text{Geodesic Equation.}\end{aligned}$$

It appears that we have derived four equations of motion from stationarity under the four variations $\delta x^\mu(\lambda)$. This is not really true. Consider the variation

$$x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda) = x^\mu(\lambda + \delta\lambda) = x^\mu(\lambda').$$

The change in parameterization of th path does not affect $\int d\tau$, which is geometrical, for any $x^\mu(\lambda)$, physical or not. Thus $\delta S = 0$ in an identity, not an equation of motion, for $\delta x^\mu(\lambda) \propto dx^\mu/d\lambda$. This can be verified directly by multiplying the equation of motion by $g_{\nu\mu} dx^\nu/d\tau$:

$$g_{\nu\mu}\frac{dx^\nu}{d\tau}\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\mu\nu\rho}\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau}$$

$$= \frac{1}{2}\frac{d}{d\tau}\Bigg(\underbrace{\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}g_{\mu\nu}}_{1}\Bigg) + \underbrace{\left(\Gamma_{\mu\nu\rho} - \frac{1}{2}g_{\mu\nu,\rho}\right)}_{\frac{1}{2}g_{\mu\rho,\nu}-\frac{1}{2}g_{\nu\rho,\mu}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0,$$

as 1 is a constant and by symmetry under $\mu \leftrightarrow \nu$.

Why are we only getting 3 equations from our four unknowns? Why is the action refusing to tell us equations which determine from initial conditions the subsequent motion $x^\mu(\lambda)$? Because there is a real arbitrariness. The path $\Gamma = \{x^\mu(\lambda)\}$ is physical, but the parameterization has no physical meaning, so the physics cannot tell you how far to go along the path Γ for a given $\Delta\lambda$. This is an example of a local gauge invariance — not all of the variables used in writing the possible motions are physical. One way of handling such problems is to choose the undetermined parameter by a supplementary condition, called a choice of gauge. For example, we might require $\lambda = \tau$. In electromagnetism there is a similar problem with A_μ. The gauge invariant action cannot determine A_μ because $A'_\mu = A_\mu + \partial_\mu\Lambda(\vec{x}, t)$ is just as good, and Λ can be chosen to give no contribution at the initial time but make changes later. In electromagnetism we sometimes impose the Lorenz gauge condition $\partial_\mu A^\mu = 0$. This provides the necessary fourth equation to determine A^μ in the future from initial conditions.

There is another approach, which is to screw up the action so that it is no longer gauge invariant. For example, adding a term $(\partial_\mu A^\mu)^2$ to the lagrangian density. This will produce the usual equations plus the equation $\partial_\mu A^\mu = 0$, so it is equivalent to imposing a supplementary condition from the start. Similarly, if we write

$$S' = \int -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,d\lambda,$$

leaving out the square root, we find the geodesic equation with $d/d\lambda$ instead of $d/d\tau$. Multiplying by $g_{\nu\mu} dx^\nu/d\lambda$ tells us $\lambda \propto \tau$.

It should be said, however, that all of these approaches "fix a gauge" in an arbitrary fashion and therefore remove some of the symmetry inherent in

the physics. One must not treat an equation like $\partial_\mu A^\mu$ as a fundamental law of physics, as one can the equation $\mathbf{d}^*\mathbf{F} = {}^*\mathbf{J}$, for example.

Having derived the equation which determines how otherwise free particles move in a gravitational field, let us compare with Newton's laws for a test particle in the field of a single heavy body at rest. We must limit our attention to slow particles and weak gravitational fields, for otherwise Newton can't be right. Furthermore, it should be possible to choose our coordinates so that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \ll 1$, so $\Gamma \ll 1$. Then to first order in h, Γ and v, $t = \tau$, $u^\mu = (1, \vec{v})$, and the geodesic equation and $\vec{F} = m\vec{a} = -\vec{\nabla}\phi$ give

$$\frac{d^2x^j}{dt^2} = -\Gamma^j{}_{00} = -\partial_j\phi,$$

where ϕ is Newton's gravitional potential $\phi = -GM/r$. Assume $g_{\mu\nu}$ is independent of time. Then

$$\Gamma^j{}_{00} \approx \frac{1}{2}\left(g_{j0,0} + g_{0j,0} - g_{00,j}\right) = -\frac{1}{2}g_{00,j}$$

so $g_{00} = 1 - 2\phi$. This is the Newtonian approximation.

Consider now a stationary metric, $g_{\mu\nu}(\vec{x})$ independent of t, not necessarily weak. Consider two clocks at rest in this field. Each clock is guaranteed by the manufacturer to tick once each second of proper time regardless of acceleration (no grandfather clocks allowed). In terms of our coordinate system

$$(\Delta\tau)^2 = -g_{\mu\nu}\Delta x^\mu\,\Delta x^\nu = -g_{00}(\vec{x})\;(\Delta t)^2$$

so the coordiante interval between ticks is

$$(\Delta t)_A = \left[-g_{00}(x_A)\right]^{-1/2}, \qquad (\Delta t)_B = \left[-g_{00}(x_B)\right]^{-1/2}.$$

If A sends light signals to B each time his clock ticks, the time differences $t_{\text{received}} - t_{\text{emitted}}$ will be the same for each pulse, so B can measure on his own clock the period between ticks of A's clock. The answer is

$$T' = \frac{\Delta t_A}{\Delta t_B} = \left[\frac{g_{00}(x_B)}{g_{00}(x_A)}\right]^{1/2}$$

and the frequency of the light emitted is therefore shifted by

$$f' = f\left[\frac{g_{00}(x_A)}{g_{00}(x_B)}\right]^{1/2}.$$

We have derived this for an arbitrary stationary metric. In the Newtonian limit

$$\frac{f'}{f} = \left[\frac{1+2\phi_A}{1+2\phi_B}\right]^{\frac{1}{2}} \approx 1 + (\phi_A - \phi_B) = 1 + GM\left[\frac{1}{r_B} - \frac{1}{r_A}\right].$$

At the surface of the Sun $\phi = -2.12 \times 10^{-6}$, so for an observer at ∞, the Sun's light is red shifted by

$$\Delta f/f = +\phi_{\rm surface} = -2.12 \times 10^{-6}.$$

Note that $f'/f = 1+\phi_A-\phi_B$ agrees with our calculation based on equivalence to a rocket ship.

This gravitational red shift is best tested by dropping photons down a shaft at Harvard. General relativity has been tested thereby to an accuracy of about 1%.

Read chapter 3 of Weinberg, which we have just finished.

0.9 Parallel Transport

Consider a manifold with a vector or a 1-form defined at each point. such an object might be a physical field, which would have field equations involving derivatives of this vector quantity. How can we tell on a curved manifold whether a vector $\mathbf{V} = V^\mu(x_A)\partial_\mu$ at the point A is the same or different from $\mathbf{V}$ at B, $V^\mu(x_B)\partial_\mu$? The naïve thing would be if $V^\mu(x_A) = V^\mu(x_B)$, But the V^μ's are chart dependent and such a statement of equality of components at different points can be true for one chart and not another, and has no real meaning for the manifold. It is also the *wrong* requirement even for the simple example of a two dimensional Euclidean space in polar coordinates, for a vector in the ρ direction for $\phi = 0$ is completely different from one of the same magnitude in the ρ direction at $\phi = \pi/2$.

This is true even for A and B very near each other. By definition, all charts agree on whether two directions differ by a finite angle as $A \to B$, but not on the rate. Thus $\lim_{x_B \to x_A} V^\mu(x_B)$ is well defined, but not $\partial_\mu V^\mu(x_B)$, in the sense that it is chart-dependent.

Thus an arbitrary manifold has no means of comparing vectors at different points, unless there is an extra structure placed on the manifold telling how to move a coordinate system from point A to a nearby point B.

Let the equivalence principle help us out, by giving us a chart $C' = \{\xi^\mu\}$ of a neighborhood of the event A which is cartesian and inertial at A. Then $g'_{\alpha\beta}(x_A) = \eta_{\alpha\beta}$ and $g'_{\alpha\beta,\gamma}(x_A) = 0$. A vector $\mathbf{V} = V^\mu\partial_\mu = V'^\mu\partial'_\mu$ defined at A is parallel transported an infinitesimal distance from A by holding its coordinates fixed, because that's how one parallel transports in flat space cartesian coordinates. Thus if we have a vector field $\mathbf{V}$ and we ask what the "physical change" in $\mathbf{V}$ is along the ∂'_α direction, it is $\mathbf{V}(B) - \mathbf{V}(A)_{\rm transported} \approx \Delta\xi^\alpha\left(\partial'_\alpha V'^\beta\right)\partial'_\beta$. Let us define that to be $\Delta\xi^\alpha$ times the **covariant derivative** $D_{\alpha'}$ in the $\Delta\xi^\alpha$ direction. With Δx^μ the corresponding change in chart C's coordinates, we have

$$\begin{aligned} \Delta x^\mu D_\mu \mathbf{V} &= \Delta\xi^\alpha\left(\partial'_\alpha V'^\beta\right)\partial'_\beta \\ &= \Delta x^\mu \frac{\partial\xi^\alpha}{\partial x^\mu}\partial'_\alpha\left(V^\nu\frac{\partial\xi^\beta}{\partial x^\nu}\right)\frac{\partial x^\rho}{\partial\xi^\beta}\partial_\rho \\ \text{or}\quad D_\mu\mathbf{V} &= \partial_\mu\left(V^\nu\frac{\partial\xi^\beta}{\partial x^\nu}\right)\frac{\partial x^\rho}{\partial\xi^\beta}\partial_\rho \end{aligned}$$

$$= \left[\partial_\mu V^\rho + \underbrace{\frac{\partial^2 \xi^\beta}{\partial x^\mu \partial x^\nu}\frac{\partial x^\rho}{\partial \xi^\beta}}_{\Gamma^\rho{}_{\nu\mu}} V^\nu\right]\partial_\rho$$

Thus the components of the vector $D_\mu \mathbf{V} = (D_\mu v)^\rho\, \partial_\rho$ are

$$(D_\mu V)^\rho = \partial_\mu V^\rho + \Gamma^\rho{}_{\nu\mu} V^\nu.$$

Again, D_μ is known as the **covariant dervivative**.

The chart C was an arbitrary chart. In some other chart with coordinates x'^μ, would we have $(D_\mu \mathbf{V})^\rho$ behave like a suitable tensor, with one covariant and one contravariant index? That is, is

$$\left(D'_\mu \mathbf{V}\right)^{\rho'} \stackrel{?}{=} \frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial x^\nu}{\partial x'^\mu}\left(D_\nu \mathbf{V}\right)^\sigma$$

The left hand side is

$$\begin{aligned}(D_{\mu'}\mathbf{V})^{\rho'} &= \partial'_\mu V'^\rho + \Gamma'^\rho{}_{\mu\kappa} V'^\kappa \\ &= \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu\left(\frac{\partial x'^\rho}{\partial x^\sigma}V^\sigma\right)}_{\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x'^\rho}{\partial x^\sigma}\partial_\nu V^\sigma + \left(\frac{\partial}{\partial x'^\mu}\frac{\partial x'^\rho}{\partial x^\lambda}V^\lambda\right)} + \Gamma'^\rho{}_{\mu\kappa}\frac{\partial x'^\kappa}{\partial x^\lambda}V^\lambda\end{aligned}$$

The right hand side is

$$\frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial x^\nu}{\partial x'^\mu}\left(\partial_\nu V^\sigma + \Gamma^\sigma{}_{\nu\lambda}V^\lambda\right).$$

We find covariance if it is true that

$$\Gamma'^\rho{}_{\mu\kappa} \stackrel{?}{=} \frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\kappa}\Gamma^\sigma{}_{\nu\lambda} - \frac{\partial x^\lambda}{\partial x'^\kappa}\frac{\partial}{\partial x'^\mu}\frac{\partial x'^\rho}{\partial x^\lambda}.$$

Note
$$\begin{aligned}\frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^\lambda}{\partial x'^\kappa}\frac{\partial x'^\rho}{\partial x^\lambda}\right) &= \frac{\partial}{\partial x'^\mu}\delta^\rho_\kappa = 0 \\ &= \frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\kappa}\frac{\partial x'^\rho}{\partial x^\lambda} + \frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial}{\partial x'^\mu}\frac{\partial x'^\rho}{\partial x^\lambda}\end{aligned}$$

The second term matches our above questionable identity, which becomes

$$\Gamma'^\rho{}_{\mu\kappa} \stackrel{?}{=} \frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\kappa}\Gamma^\sigma{}_{\nu\lambda} + \frac{\partial x'^\kappa}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\kappa}.$$

But this is true, as verified from the definition

$$\Gamma^\lambda{}_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$

in Weinberg 4.5.2.

We have gone to great lengths to define the covariant derivative of a vector, which is nontrivial because the basis vectors may change from point to point. There were no such difficulties for a scalar, as the scalar did not require basis vectors. Thus $D_\mu f = \partial_\mu f$. For forms we must again worry about a basis, but we can take a shortcut if we use Leibniz product rule, $D_\mu(AB) = D_\mu(A)B + AD_\mu(B)$ which must hold for any derivative (and in particular it holds for $\partial/\partial\xi^\alpha$). Let $\mathbf{A} = A_\mu \mathbf{d}x^\mu$ be a 1-form which we wish to covariantly differentiate. With $\mathbf{V}$ and arbitrary vector,

$$\begin{aligned}D_\mu\langle \mathbf{A}|\mathbf{V}\rangle = \partial_\mu\langle \mathbf{A}|\mathbf{V}\rangle &= \langle D_\mu \mathbf{A}|\mathbf{V}\rangle + \langle \mathbf{A}|D_\mu \mathbf{V}\rangle \\ = \partial_\mu\left(A_\nu V^\nu\right) &= = \left(D_\mu \mathbf{A}\right)_\nu V^\nu + A_\rho\left(\partial_\mu V^\rho + \Gamma^\rho{}_{\nu\mu}V^\nu\right).\end{aligned}$$

Thus

$$\left(D_\mu \mathbf{A}\right)_\nu = \partial_\mu A_\nu - \Gamma^\rho{}_{\nu\mu}A_\rho.$$

The rules for an arbitrary tensor can be found by considering tensor products of vectors and 1-forms. We find

$$\begin{aligned}(D_\mu T)^{\nu_1\ldots\nu_r}{}_{\rho_1\ldots\rho_s} &= \partial_\mu T^{\nu_1\ldots\nu_r}{}_{\rho_1\ldots\rho_s} + \sum_{i=1}^r \Gamma^{\nu_i}{}_{\alpha\mu}T^{\nu_1\ldots\nu_{i-1}\alpha\nu_{i+1}\ldots\nu_r}{}_{\rho_1\ldots\rho_s} \\ &\quad - \sum_{i=1}^s \Gamma^\alpha{}_{\rho_i\mu}T^{\nu_1\ldots\nu_r}{}_{\rho_1\ldots\rho_{i-1}\alpha\rho_{i+1}\ldots\rho_s}.\end{aligned}$$

The relationship between forms and vectors we just preserved in our definition of D on a form has nothing to do with the metric. But another connection we would like to have is that parallel transport of a pair of vectors should not change their inner product $\mathbf{g}(\mathbf{u},\mathbf{v})$. Thus $\partial_\mu \mathbf{g}(\mathbf{u},\mathbf{v}) = D_\mu \mathbf{g}(\mathbf{u},\mathbf{v}) = 0$ if $D_\mu \mathbf{u} = 0$ and $D_\mu \mathbf{v} = 0$. But

$$D_\mu \mathbf{g}(\mathbf{u},\mathbf{v}) = \left(\mathbf{D}_\mu \mathbf{g}\right)(u,v) + \mathbf{g}\left(\mathbf{D}_\mu \mathbf{u}, v\right) + \mathbf{g}\left(\mathbf{u}, \mathbf{D}_\mu \mathbf{v}\right),$$

and the last two terms are zero, so we must have

$$D_\mu \mathbf{g} = 0.$$

To check this, evaluate

$$\left(D_\mu \mathbf{g}\right)_{\rho\sigma} = g_{\rho\sigma,\mu} - \Gamma^\lambda{}_{\rho\mu} g_{\lambda\sigma} - \Gamma^\lambda{}_{\sigma\mu} g_{\lambda\rho}$$

by our general relation for a tensor, so

$$\left(D_\mu \mathbf{g}\right)_{\rho\sigma} = g_{\rho\sigma,\mu} - \Gamma_{\sigma\rho\mu} - \Gamma_{\rho\sigma\mu} = 0,$$

which is true (see notes p 46), so all is well.

Note: As $\mathbf{D}g = 0$, $\mathbf{D}$ commutes with raising and lowering indices! That is important, *e.g.*

$$g^{\mu\nu}\left(\mathbf{D}_\rho \underbrace{\mathbf{A}}_{1\text{-form}}\right)_\nu = \left(\mathbf{D}_\rho \underbrace{\mathbf{A}}_{\text{vector}}\right)^\mu .$$

Our definition of covariant derivative assumed the vector or scalar or 1-form was a field defined in the neighborhood of the event. Sometimes there are quantities only defined on, for example, a path. The velocity of a particle as it moves along its world-line is an example. $\mathbf{u}$ is simply not defined except along the path, and neither is, say, the spin of the particle $\mathbf{S}$. But we can define a convariant derivative along the path as it would be were $\mathbf{S}$ defined everywhere

$$\left(\frac{D}{D\lambda}\mathbf{S}\right)^\nu = \text{“}\left(\frac{dx^\mu}{d\lambda}D_\mu S\right)^\nu\text{”} = \text{“}\frac{dx^\mu}{d\lambda}\left(\frac{\partial S^\nu}{\partial x^\mu} + \Gamma^\nu{}_{\rho\mu} S^\rho\right)\text{”} = \frac{dS^\nu}{d\lambda} + \Gamma^\nu{}_{\rho\mu} S^\rho \frac{dx^\mu}{d\lambda}.$$

The last expression is well-defned entirely along the path of the particle, even though the expressions in quotes are not.

Recall from homework $\vec\nabla f \sim \mathbf{d}f$ and doesn't require any knowledge of $\mathbf{g}$. Similarly $\vec\nabla \times \vec A \sim \mathbf{dA}$ doesn't depend on $\mathbf{g}$ or Γ. But $\vec\nabla \cdot \vec A = {}^*\mathbf{d}\,{}^*\mathbf{A}$, and the * requires the use of $\epsilon_{\mu\nu\rho\sigma} = \sqrt{g}\,[\mu\nu\rho\sigma]$, where it is $[\mu\nu\rho\sigma]$, not $\epsilon_{\mu\nu\rho\sigma}$, which is a constant ($\pm 1$ or 0). Thus if $\mathbf{A} = A_\mu \mathbf{d}x^\mu$,,

$$\begin{aligned}
{}^*\mathbf{A} &= \frac{1}{3!} A^\mu \sqrt{g} [\mu\rho\sigma\kappa] \mathbf{d}x^\rho \wedge \mathbf{d}x^\sigma \wedge \mathbf{d}x^\kappa \\
\mathbf{d}\,{}^*\mathbf{A} &= \frac{1}{3!} \partial_\nu \left(A^\mu \sqrt{g}\right) [\mu\rho\sigma\kappa] \mathbf{d}x^\nu \wedge \mathbf{d}x^\rho \wedge \mathbf{d}x^\sigma \wedge \mathbf{d}x^\kappa \\
\text{and}\quad {}^*\mathbf{d}\,{}^*\mathbf{A} &= \frac{1}{3!} \partial_\nu \left(A^\mu \sqrt{g}\right) g^{-1/2} [\nu\rho\sigma\kappa][\mu\rho\sigma\kappa] = g^{-1/2} \partial_\nu \left(A^\mu \sqrt{g}\right)
\end{aligned}$$

That is perhaps not what you expected ($\partial_\mu A^\mu$?). But it is the covariant derivative of the vector $\mathbf{A}$, contracted to form a divergence,

$$D_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu{}_{\nu\mu} A^\nu,$$

$$\begin{aligned}
\text{as}\qquad \Gamma^\mu{}_{\nu\mu} &= \frac{1}{2} g^{\mu\rho} \left(g_{\nu\rho,\mu} + g_{\mu\rho,\nu} - g_{\nu\mu,\rho}\right) = \frac{1}{2} g^{\mu\rho} g_{\mu\rho,\nu} = \frac{1}{2} \operatorname{Tr} G^{-1} \partial_\nu G \\
&= \frac{1}{2} \operatorname{Tr} \partial_\nu \ln \det G = \partial_\nu \ln \left(g^{1/2}\right) = g^{-1/2} \partial_\nu g^{1/2}
\end{aligned}$$

(where the matrix $G = g_{\cdot\cdot}$.)

Thus $\qquad D_\mu A^\mu = g^{-1/2} \partial_\nu \left(A^\mu \sqrt{g}\right)$.

If we used D_μ for the divergence, why not for the curl? We did, but it made no difference,

$$\vec\nabla \times \vec A \sim D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - \left(\underbrace{\Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu}}_{0}\right) A_\rho,$$

so the Γ dependence falls out of the antisymmetric part of the covariant derivative of a 1-form. Define the tensor product of two 1-forms and therein lies (if $\mathbf{A}$ is a 1-form)

$$\mathbf{d}x^\mu \otimes D_\mu \mathbf{A} = \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu \left(\partial_\mu A_\nu - \Gamma^\rho{}_{\nu\mu} A_\rho\right).$$

The antisymmetric part is just $\mathbf{dA}$, but the symmetric part is dependent on the connection coefficients. Similarly $\mathbf{dF} \sim D_{[\mu} F_{nu\rho]}$, and the Γ drops out.

Restatement:

Let us suppose we have a physical system involving a field ψ which takes values in a vector space, so that in some particular basis we have $\psi^a, a = 1, \cdots, N$. Let us also suppose that the phuysics is invariant under a group of transformations o fthe basis, or under

$$\psi'^a = \left(e^{-i\theta L}\right)^a{}_b \psi^b \tag{8}$$

made independently at each spacetime point. then if any derivatives are to enter the theory at all, there must be some additional structure. Let us assume a kind of equivalence principle: at any one point $\mathcal{P}$ of spacetime it is possible to find a set of bases $e_a(x)$ such that, at $\mathcal{P}$, the physics is described by a Lagrangian $\mathcal{L}(\psi, \partial_\mu\psi)$ with no other fields (analogous to the laws of special relativity with no gravitational fields). Then in any other basis, the Lagrangian must be described by

$$\mathcal{L}(\psi', D'_\mu\psi')$$

where the relationship between the bases (8) also holds for

$$\begin{aligned} D'_\mu\psi' &= \left((e^{-i\theta L}\right)^a{}_b \partial_\mu\psi^b = \left(e^{-i\theta L}\right)^a{}_b \partial_\mu \left(e^{i\theta L}\psi'\right)^b \\ &= \partial_\mu\psi'^{\,a} + \left(e^{-i\theta L}\right)^a{}_b \partial_\mu \left(e^{i\theta L}\right)^b{}_c \psi'^{\,c}, \end{aligned}$$

or $D_\mu = \mathbb{I}\partial_\mu + e^{-i\theta L}\partial_\mu e^{i\theta L}$ is a matrix acting on the vector space of the ψ's. Define

$$A^a{}_{c\,\mu} = e^{-i\theta L}\,\partial_\mu\, e^{i\theta L} = \mathbf{A}_\mu.$$

Note that although $e^{i\theta L}$ connects two bases at the same point, the one for which the "inertial" frame has no A, one bases that inertial frame's and the other an arbitrary, general basis, the $\mathbf{A}$ refers only to the general basis, but in a sense at neighboring points. It defines parallel transport in the vector space of the ψ's.

0.10 Electromagnetism in Flat Space

In quantum field theory, a charged particle is described by a complex field $\psi(x)$. Only the magnitude of the field has direct physical interpretation, so therei is no inherent meaning to what is the "real direction" for ψ. It might be easier to think of ψ as a two component real vector $\psi = \psi_1 + i\psi_2$. All objects, *e.g.* $\mathcal{L}$, J_μ, *etc.*, which one could write in therms of $\psi^*\psi$ can be rewritten as contractions of 2-vectors, $(\psi_1, \psi_2)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$. There is no physics to which is the 1 and whcih is the 2 direction, and under a rotation $\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to \Psi' = e^{i\phi\sigma_y}\Psi$, such dot products don't change and the physics is invariant under this global gauge transformation.

Suppose, however, we require physics to be expressible in terms which permit a rotation in the $(1,2)$ coordinate system independently at each point in space-time. Then we have the same trouble defining the derivative that we had in gravity. $\psi_1(A) - \psi_1(B)$ has no intrinsic meaning, and we must define a parallel transport of the basis vectors on $(1,2)$ space.

Suppose it is possible, at a point, to choose a coordinate system such that $D_\mu\Psi^a = \partial_\mu\Psi^a$, $a = 1, 2$. In some other basis $\Psi'^{\,a} = \left(e^{-i\phi(x)\sigma_y}\right)^a{}_b \Psi^b$, and, because $D_\mu\Psi$ should rotate like Ψ,

$$\begin{aligned} (D_\mu\Psi)'^{\,a} &= \left(e^{-i\phi(x)\sigma_y}\right)^a{}_b (D_\mu\Psi)^b \\ &= \left(e^{-i\phi(x)\sigma_y}\right)^a{}_b \partial_\mu\left[\left(e^{i\phi(x)\sigma_y}\right)^b{}_c \Psi'^{\,c}\right] \\ &= \partial_\mu\Psi'^{\,a} + (\partial_\mu\phi)\,(i\sigma_y)^a{}_c\,\Psi'^{\,c} \\ &=: \partial_\mu\Psi'^{\,a} + A^a{}_{c\,\mu}\Psi'^{\,c}. \end{aligned}$$

The A field is seen to be analogous to the connection coefficient Γ. In our simple E-M system A is always proportional to $(i\sigma_y)^a{}_c$, wo we generally write

$$A^a{}_{c\,\mu} =: q(i\sigma_y)^a{}_c A_\mu$$

where q is the electric charge of the particle.

Start with a Lagrangian $\mathcal{L}_1(\Psi, \partial_\mu\Psi)$ which does not have this "local gauge invariance", but is invariant under a global transformation $\Psi \to e^{-ic\sigma_y}\Psi$, for c a constant. Then let

$$\mathcal{L} = \mathcal{L}_1(\Psi, D_\mu\Psi)$$

be a different (A field dependent) lagrangian. It will describe the interactions of the charges with the electromagnetic field via the "principle of minimal substitution"

$$\partial_\mu \to \partial_\mu + q(i\sigma_y)A_\mu.$$

Another way to write this is to define the matrix $\mathbf{A}_\mu := g(i\sigma_y)\,A_\mu$, and then the covariant derivative is a matrix operator $D_\mu = 1\!\!1\partial_\mu + \mathbf{A}_\mu$. This is usually written in the complex form,

$$\partial_\mu \to D_\mu = \partial_\mu - iqA_\mu.$$

The new $\mathcal{L}$ is invariant under $\Psi \to e^{-i\sigma_y\phi(x)}\Psi$ provided we simultaneously make the gauge transform

$$\mathbf{A}_\mu \to \mathbf{A}_\mu - \frac{i}{q}\,\sigma_y\,\partial_\mu\phi.$$

Consider as an example

$$\mathcal{L}_1 = \frac{1}{2}\Phi^*_{,\mu}\Phi^{,\mu} - \frac{m^2}{2}\Phi^*\Phi = \sum_a \left\{\frac{1}{2}\phi^a_{,\mu}\phi^{a\,,\mu} - \frac{m^2}{2}\Phi^a\Phi^a\right\}.$$

The Euler-Lagrange equations $\dfrac{\partial\mathcal{L}_1}{\partial\phi^a} = \partial_\mu\dfrac{\partial\mathcal{L}_1}{\partial\phi^a_{,\mu}}$ gives

$$\partial_\mu\partial^\mu\phi^a + m^2\phi^a = 0$$

which is the Klein-Gordon equation for a scalar field. The lagrangian $\mathcal{L}_1$ is invariant under $\delta\phi^a = (i\sigma_y)^a{}_b\,\phi^b$ and the corresponding Noether conserved current is

$$J^\mu = \frac{\partial\mathcal{L}_1}{\partial\phi^a_{,\mu}}\delta\phi^a = \phi^{a\,,\mu}(-i\sigma_y)^a{}_b\phi^b,$$

which is conserved as $\partial_\mu J^\mu = -m^2\phi(-i\sigma_y)\phi + \phi^{a\,,\mu}(-i\sigma_y)^a{}_b\phi^b_{,\mu} = 0$ by the antisymmetry[9] of σ_y.

Now consider the modified lagrangian

$$\mathcal{L}(\phi, A) = \frac{1}{2}\left(D_\mu\phi\right)^a\left(D^\mu\phi\right)^a - \frac{m^2}{2}\Phi^a\Phi^a$$

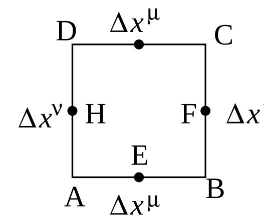
[9]Note moving the a index up or down makes no difference.

which gives the equation of motion $D_\mu D^\mu\Phi + m^2\Phi = 0$ and the current $J^\mu = \left(D_\mu\phi\right)^a(-i\sigma_y)^a{}_b\phi^b$. J^μ is now unchanged by a local gauge transformation, so it is a scalar with respect to the gauge group. (*i.e.* independent of the two dimensional rotations). Thus

$$\begin{aligned}\partial_\mu J^\mu &= D_\mu J^\mu = \left(D_\mu D^\mu\phi\right)(-i\sigma_y)\phi + D^\mu\phi(-i\sigma_y)D_\mu\phi \\ &= -m^2\phi(-i\sigma_y)\phi + D^\mu\phi(-i\sigma_y)D_\mu\phi = 0,\end{aligned}$$

just as before. But we now have the electromagnetic field included.

Consider a little rectangle and an electromagnetic field A_μ throughout the region, as well as a two component field Φ. Suppose I parallel transport Φ from A to B to C to D and then back to A. Will I get the same Φ that I started with? Let's try.



$$\Delta x^\mu D_\mu\Phi \sim \Delta x^\mu\left(\partial_\mu\Phi + \mathbf{A}_\mu(E)\Phi\right) = 0$$

so $\partial_\mu\Phi = -\mathbf{A}_\mu\Phi$.

$$\begin{aligned}\text{Now}\quad \Phi_B - \Phi_A &= -\Delta x^\mu\mathbf{A}_\mu(E)\Phi(E) + \mathcal{O}\left((\Delta x)^3\right). \\ \text{Similarly}\quad \Phi_C - \Phi_B &= -\Delta x^\nu\mathbf{A}_\nu(F)\Phi(F), \\ \Phi_D - \Phi_C &= \Delta x^\mu\mathbf{A}_\nu(G)\Phi(G), \\ \Phi'_A - \Phi_D &= \Delta x^\nu\mathbf{A}_\nu(H)\Phi(H).\end{aligned}$$

Let's evaluate this to second order in Δ's:

$$\begin{aligned}\Phi'_A - \Phi_A &= \Delta x^\mu\,\mathbf{A}_\mu\Phi\big|_E^G - \Delta x^\nu\,\mathbf{A}_\nu\Phi\big|_H^F \\ &\approx \Delta x^\mu\,\Delta x^\nu\left[\partial_\nu\left(\mathbf{A}_\mu\Phi\right) - \partial_\mu\left(\mathbf{A}_\nu\Phi\right)\right] \\ &= -\Delta x^\mu\,\Delta x^\nu\,\left\{\left(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu\right)\Phi + \mathbf{A}_\mu\mathbf{A}_\nu\Phi - \mathbf{A}_\nu\mathbf{A}_\mu\Phi\right\} \\ &= -\Delta x^\mu\,\Delta x^\nu\,\mathbf{F}_{\mu\nu}\Phi,\end{aligned}$$

where

$$\mathbf{F}_{\mu\nu} := \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + \mathbf{A}_\mu\mathbf{A}_\nu - \mathbf{A}_\nu\mathbf{A}_\mu.$$

We do not come back to what we started. This is true in the $(1,2)$ gauge space for Φ in electromagnetism, and it is true for a 4-vector in curved space as well. I have actually been more careful than you might have expected, because the **A**'s are matrices so one should be careful to keep their orders intact. For electromagnetism it doesn't matter, as all the **A**'s are proportional to σ_y and so they commute, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. But we might think of a more general cse, where we would have the **A**'s a nontrivial set of matrices, and **F** too, with

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu] = [D_\mu, D_\nu]\,.$$

This is Yang-Mills field theory, also known as non-Abelian gauge field theory. Special cases include

(a) the Salam-Weinberg unified model of the weak and electromagnetic interactions, for which they got the Nobel Prize in 1979. Here the **A**'s are 2×2 unitary matrices representing the photon, the $W^\pm$ and the Z^0 particles.

(b) Quantum chromodynamics (QCD) the generally accepted theory of the strong interactions. The Φ's have three components called colors (red, green and blue quarks), and the **A**'s are unitary 3×3 matrices with determinant 1 acting on these three component vectors. This group is $\mathrm{SU}(3)_{\mathrm{color}}$, and the **A**'s ar gluons.

(c) Grand unified gauge field theories postulate that the **A**'s are bigger matrices forming a Lie algebra. Various possibilities considered have been SU(5), O(10), O(14), O(18), O(22). Each contains the $\mathrm{SU}(3)_{\mathrm{color}} \times U(2)_{\mathrm{SW}}$. some of the **A**'s are "baseballs", some cause the proton to decay. These theories are still very speculative.

(d) Although it is not quite a Yang-Mills theory because of features we have not yet discussed, gravity has the same property. If Φ is a vector, $\mathbf{A}_\mu$ is a 4×4 matrix with components $(\mathbf{A}_\mu)^\rho{}_\sigma = \Gamma^\rho{}_{\sigma\mu}$. It is not usually called **A**, of course. The $\mathbf{F}_{\mu\nu}$ in this case is called $R_{\mu\nu}$, which in terms of Γ becomes

$$(R_{\mu\nu})^\alpha{}_\beta = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\gamma\mu}\Gamma^\gamma{}_{\beta\nu} - \Gamma^\alpha{}_{\gamma\nu}\Gamma^\gamma{}_{\beta\mu}.$$

Standard notation for this is $R^\alpha{}_{\beta\mu\nu}$ in MTW, or $-R^\alpha{}_{\beta\mu\nu}$ in Weinberg.

R is actually a tensor, unlike Γ. Γ is not a tensor because the matrix indices (the first two) refer, in a sense, to different points. Its change under

change of chart therefore depends not only on the way the coordinates change at the point in question but also on their derivatives. the matrix indices of R, however, refer to the same point, for the vector has been brought back to the starting point. This is the reason, but the proof of the pudding is to use the rules for changes of chart for Γ and discover that the inhomogeneous terms cancel.

Weinberg shows R is the only tensor which

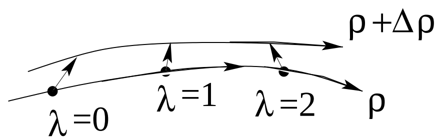
1. depends only on **g** and its first and second derivatives, and

2. is linear in the second derivatives.

It is called the curvature tensor, or in MTW, "Riemann".

0.11 Geodesic Deviation

Consider a set of trajectories through spacetime, parameterized by ρ, each of which is a curve parameterized by a parameter λ, separated by a small distance. At each fixed λ, we may consider $\mathbf{n} = \Delta\mathcal{P}|_\lambda$ as a vector $\Delta\rho\dfrac{\partial\mathcal{P}}{\partial\rho} = \Delta\rho\dfrac{\partial x^\mu}{\partial\rho}\dfrac{\partial\mathcal{P}}{\partial x^\mu} = \Delta\rho\dfrac{\partial x^\mu}{\partial\rho}\partial_\mu$, where x^μ is some chart. Then MTW p219 calls $n^\mu = \Delta\rho\dfrac{\partial x^\mu}{\partial\rho}$.



Let us ask how $\mathbf{n}$ develops as we move along the trajectories, assuming each of the trajectories obeys the law of geodesic transport:

$$\frac{Du^\nu}{D\lambda} = \left.\frac{du^\nu}{d\lambda}\right|_\rho + \Gamma^\nu{}_{\rho\sigma}u^\rho u^\sigma = 0 \qquad \text{with} \quad u^\nu = \left.\frac{\partial x^\nu}{\partial\lambda}\right|_\rho .$$

$$\text{ow} \quad \left.\frac{\partial n^\mu}{\partial\lambda}\right|_\rho = \Delta\rho\frac{\partial^2 x^\mu}{\partial\lambda\,\partial\rho} = \delta\rho\,\left.\frac{\partial u^\mu}{\partial\rho}\right|_\lambda .$$

Taking $\delta\rho\dfrac{\partial}{\partial\rho}$ of the geodesic equation, we have

$$\frac{\partial^2 n^\mu}{\partial\lambda^2} + \underbrace{\delta\rho\,\frac{\partial}{\partial\rho}\Gamma^\mu{}_{\rho\sigma}\Big|_\lambda}_{\delta\rho\frac{\partial x^\alpha}{\partial\rho}\Gamma^\mu{}_{\rho\sigma,\alpha}n^\alpha\Gamma^\mu{}_{\rho\sigma,\alpha}} u^\rho u^\sigma + 2\Gamma^\mu{}_{\rho\sigma}u^\rho\frac{\partial n^\sigma}{\partial\lambda} = 0 \qquad (9)$$

Let us use this to evaluate the second derivative along the free-falling path of $\mathbf{n}$,

$$\begin{aligned}
\frac{D}{D\lambda}\frac{Dn^\mu}{D\lambda} &= \frac{D}{D\lambda}\left(\frac{\partial n^\mu}{\partial\lambda} + \Gamma^\mu{}_{\rho\sigma}n^\rho u^\sigma\right)\\
&= \frac{\partial}{\partial\lambda}\left(\frac{\partial n^\mu}{\partial\lambda} + \Gamma^\mu{}_{\rho\sigma}n^\rho u^\sigma\right) + \Gamma^\mu{}_{\alpha\beta}u^\beta\left(\frac{\partial n^\alpha}{\partial\lambda} + \Gamma^\alpha{}_{\rho\sigma}n^\rho u^\sigma\right)\\
\frac{D^2 n^\mu}{D\lambda^2} &= \frac{\partial^2 n^\mu}{\partial\lambda^2} + \Gamma^\mu{}_{\rho\sigma,\beta}n^\rho u^\sigma u^\beta + 2\Gamma^\mu{}_{\rho\sigma}\frac{\partial n^\rho}{\partial\lambda}u^\sigma + \Gamma^\mu{}_{\rho\sigma}n^\rho \underbrace{\frac{\partial u^\sigma}{\partial\lambda}}_{-\Gamma^\sigma{}_{\beta\alpha}u^\beta u^\alpha}
\end{aligned}$$

$$\begin{aligned}
&\quad +\Gamma^\mu{}_{\alpha\beta}\Gamma^\alpha{}_{\rho\sigma}n^\rho u^\beta u^\sigma\\
0 &= \frac{\partial^2 n^\mu}{\partial\lambda^2} + \Gamma^\mu{}_{\sigma\beta,\rho}n^\rho u^\sigma u^\beta + 2\Gamma^\mu{}_{\rho\sigma}\frac{\partial n^\rho}{\partial\lambda}u^\sigma\\
\text{so} \quad \frac{D^2 n^\mu}{D\lambda^2} &= \left[\Gamma^\mu{}_{\rho\sigma,\beta} - \Gamma^\mu{}_{\sigma\beta,\rho} + \Gamma^\mu{}_{\beta\alpha}\Gamma^\alpha{}_{\rho\sigma} - \Gamma^\mu{}_{\rho\alpha}\Gamma^\alpha{}_{\beta\sigma}\right] n^\rho u^\beta u^\sigma
\end{aligned}$$

$$\text{and} \quad \frac{D^2 n^\mu}{D\lambda^2} + R^\mu{}_{\sigma\rho\beta}n^\rho u^\beta u^\sigma = 0.$$

This gives an interesting piece of information.

In the freely falling inertial chart $\frac{D}{D\lambda} = \frac{d}{dt}$ if we take $\lambda = \tau$, the spatial components give the rule for the acceleration of a particle at separation $\vec{n}$ at rest

$$\left(\frac{d^2\vec{n}}{dt^2}\right)^i = -R^i{}_{0j0}n^j.$$

Symmetries of R

We have seen that $R^\alpha{}_{\beta\mu\nu} = \langle\omega^\alpha|[D_\mu, D_\nu]\,e_\beta\rangle$, so R is antisymmetric in the last two indices. Are there other symmetries? If so, because R is a tensor, they must be true in any frame, so we will work in the local inertial frame. Recall that in that frame, $g_{\mu\nu,\rho}$ and Γ vanish, while $g_{\mu\nu} = \eta_{\mu\nu}$, so

$$\begin{aligned}
R^\alpha{}_{\beta\mu\nu} &= \partial_\mu\Gamma^\alpha{}_{\beta\nu} - (\mu\leftrightarrow\nu)\\
&= \frac{1}{2}\partial_\mu\left[g^{\alpha\rho}\left(g_{\nu\rho,\beta} + g_{\beta\rho,\nu} - g_{\beta\nu,\rho}\right)\right] - (\mu\leftrightarrow\nu)\\
&= g^{\alpha\rho}\Big[g_{\nu\rho,\beta\mu} + \underbrace{g_{\beta\rho,\nu\mu}}_{\to 0 \text{ under } \mu\leftrightarrow\nu} - g_{\beta\nu,\rho\mu}\Big] - (\mu\leftrightarrow\nu)\\
\text{so} \quad R_{\rho\beta\mu\nu} &= \frac{1}{2}\left[g_{\nu\rho,\beta\mu} - g_{\beta\nu,\rho\mu}\right] - (\mu\leftrightarrow\nu)\\
&= \frac{1}{2}\Big[g_{\nu\rho,\beta\mu} - (\beta\leftrightarrow\rho)\Big] - (\mu\leftrightarrow\nu).
\end{aligned}$$

So R is also antisymmetric on the first two indices when the first is lowered. If we interchange $\mu\leftrightarrow\rho$ and $\nu\leftrightarrow\beta$ we get $\frac{1}{2}\Big[g_{\beta\mu,\nu\rho} - (\mu\leftrightarrow\nu)\Big] - (\beta\leftrightarrow\rho) = \frac{1}{2}\Big[g_{\rho\nu,\mu\beta} - (\mu\leftrightarrow\nu)\Big] - (\beta\leftrightarrow\rho)$, which is the same, so

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} = -R_{\beta\alpha\mu\nu} = +R_{\mu\nu\alpha\beta}.$$

This is then true in any chart.

Now consider

$$R^\alpha{}_{[\mu\nu\rho]} = \frac{1}{3}\left(R^\alpha{}_{\mu\nu\rho} + R^\alpha{}_{\nu\rho\mu} + R^\alpha{}_{\rho\mu\nu}\right]$$
$$\underset{\text{inertial chart}}{\Longrightarrow} \frac{1}{6} g^{\alpha\sigma}\Big[\{(g_{\sigma\rho,\mu\nu} - (\mu \leftrightarrow \sigma)) - (\nu \leftrightarrow \rho)\}$$
$$+(\mu \to \nu \to \rho \to \mu) + (\mu \to \rho \to \nu \to \mu)\Big].$$

There are 12 terms in the bracket. In obvious shorthand, they are

$$\underbrace{\sigma\rho,\mu\nu}_{(3)} - \underbrace{\mu\rho,\sigma\nu}_{(4)} - \underbrace{\sigma\nu,\mu\rho}_{(2)} + \underbrace{\mu\nu,\sigma\rho}_{(1)} \quad + \underbrace{\sigma\mu,\nu\rho}_{(5)} - \underbrace{\nu\mu,\sigma\rho}_{(1)} - \underbrace{\sigma\rho,\nu\mu}_{(3)} + \underbrace{\nu\rho,\sigma\mu}_{(6)}$$
$$+ \underbrace{\sigma\nu,\rho\mu}_{(2)} - \underbrace{\sigma\nu,\rho\mu}_{(6)} - \underbrace{\sigma\mu,\rho\nu}_{(5)} + \underbrace{\rho\mu,\sigma\nu}_{(4)} = 0,$$

by cancellation in pairs as shown. Thus $R^\alpha{}_{[\mu\nu\rho]} = 0$ in any chart.

Bianchi Identity

$$R^\alpha{}_{\beta\mu\nu} = \langle\omega^\alpha|[D_\mu, D_\nu]\, e_\beta\rangle.$$

Let us parallel transport ω and e so $D_\rho e_\alpha = 0$ and $D_\rho\omega^\alpha = 0$. Then

$$(D_\rho R)^\alpha{}_{\beta\mu\nu} = \langle\omega^\alpha|D_\rho\,[D_\mu, D_\nu]\, e_\beta\rangle = \langle\omega^\alpha|[D_\rho, [D_\mu, D_\nu]]\, e_\beta\rangle.$$

Defining $R^\alpha{}_{\beta\mu\nu;\rho} := D_\rho R^\alpha{}_{\beta\mu\nu}$ [we will use ; as the covariant derivative for any tensor], we have

$$R^\alpha{}_{\beta\mu\nu;\rho} + R^\alpha{}_{\beta\nu\rho;\mu} + R^\alpha{}_{\beta\rho\mu;\nu} = \langle\omega^\alpha|\,\mathcal{O}\,|e_\beta\rangle\,,$$

the matrix element of

$$\mathcal{O} = [D_\rho, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]]\,.$$

But the D's are associative operators so the Jacobi identity assures us this operator is zero, which gives the

$$\text{Bianchi Identity:} \qquad R^\alpha{}_{\beta[\mu\nu;\rho]} = 0.$$

[The corresponding statement in Yang-Mills in flat spacetime is $D_{[\rho}F_{\mu\nu]} = 0$, which for an Abelian theory (E&M) reduces to $\partial_{[\rho}F_{\mu\nu]} = 0$, the homogeneous pair of Maxwell's equations.

More on the Equivalence Principle

Let $T^{\mu\nu}$ be the stress-energy tensor of matter (that is, no gravitational contribution to energy density, *etc.*), given by special relativity as a function of the fields, for example, of photons, charged particles, *etc.*. We know if we include all such matter and if there is no gravity, $\partial_\mu T^{\mu\nu} = 0$. This must still be true in the local inertial frame even if there is gravity. To make it a statement independent of chart, note that in the inertial frame $D_\mu = \partial_\mu$, so $D_\mu T^{\mu\nu} = 0$. Similarly for the electromagnetic current

$$D_\mu J^\mu = 0 = \partial_\mu J^\mu + \Gamma^\mu{}_{\nu\mu} J^\nu = g^{-1/2}\partial_\mu\left(g^{1/2}J^\mu\right).$$

This changed form for the divergence raises the question of whether charge is conserved! In special relativity we write $Q = \int J^0 d^3V$ and use $\partial_0 J^0 = \vec\nabla\cdot\vec J$ and
$int\vec\nabla\cdot\vec J d^3V = \int_S \to 0$ to show $dQ/dt = 0$. We can write this expression in a suitable chart independent way

$$Q = \int_V J^0 dx^1\wedge dx^2\wedge dx^3 = \int_V \epsilon(\mathbf{J},,,)$$

The last expression is entirely geometrical. in an arbitrary frame it reduces to

$$Q = \frac{1}{3!}\int_V \epsilon_{\mu\nu\rho\sigma} J^\mu dx^\nu\wedge dx^\rho\wedge dx^\sigma = \frac{1}{3!}\int_V \sqrt{g}\,[\mu\nu\rho\sigma]\, J^\mu dx^\nu\wedge dx^\rho\wedge dx^\sigma.$$

Let us choose V (which can be an arbitrary spacelike hypersurface) to be $t =$ constant, so

$$Q = \int \sqrt{g}\, J^0\, dx^1\wedge dx^2\wedge dx^3.$$
$$\frac{dQ}{dt} = \int_V \partial_0\left(\sqrt{g}J^0\right) d^3V = -\int \vec\nabla\cdot\left(\sqrt{g}\vec J\right) d^3V \to 0,$$

so Q is indeed conserved.

What about energy and momentum?

$$D_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\rho\mu}T^{\rho\nu} + \Gamma^\nu{}_{\rho\mu}T^{\mu\rho} = g^{-1/2}\partial_\mu\left(T^{\mu\nu}\sqrt{g}\right) + \Gamma^\nu{}_{\rho\mu}T^{\mu\rho}.$$

The first term is just what's needed to make

$$P^\mu = \int \sqrt{g}\, T^{0\mu}\, d^3V$$

conserved, but the second term breaks the conservation. This is because the gravitational force changes the momentum of the matter.

We have already discussed the form of Maxwell's laws in a geometrical form $\mathbf{d}{}^*\mathbf{F} = {}^*\mathbf{J}$, which can be written

$$\mathbf{d}\left(\sqrt{g}F^{\mu\nu}\right) \wedge \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta [\mu\nu\alpha\beta] = [\mu\nu\alpha\beta]\, \sqrt{g}\, J^\mu\, \mathbf{d}x^\nu \wedge \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta$$

or $\left(\sqrt{g}F^{\mu\nu}\right)_{,\nu} = \sqrt{g}J^\mu$.

$$\text{But} \qquad \begin{aligned} D_\nu F^{\mu\nu} &= \partial_\nu F^{\mu\nu} + \underbrace{\Gamma^\mu{}_{\rho\nu} F^{\rho\nu}}_{0} + \Gamma^\nu{}_{\rho\nu} F^{\mu\rho} \\ &= g^{-1/2}\partial_\nu\left(\sqrt{g}F^{\mu\nu}\right) = J^\mu. \end{aligned}$$

Could we have started with an equation for A in special relativity and used the equivalence principle? We start with

$$J^\mu = F^{\mu\nu}{}_{,\nu} = -A^{\mu,\nu}{}_{,\nu} + A^{\nu,\mu}{}_{,\nu}.$$

I can write this covariantly as

$$J^\mu = -A^{\mu;\nu}{}_{;\nu} + A^{\nu;\mu}{}_{;\nu} = -D_\nu D^\nu A^\mu + D_\nu D^\mu A^\nu,$$

but I could also write it in flat space as

$$J^\mu = -A^{\mu,\nu}{}_{,\nu} + A^\nu{}_{,\mu}{}^{,\nu} \Longrightarrow J^\mu \stackrel{?}{=} -D_\nu D^\nu A^\mu + D_\nu D^\mu A^\nu.$$

Are they both correct? The difference is $0 \stackrel{?}{=} \left[D_\nu, D^\mu\right] A^\nu = R^\nu{}_{\rho\nu}{}^\mu A^\rho$. We define, in general, the

$$\text{Ricci tensor:} \qquad R_{\mu\nu} := R^\alpha{}_{\mu\alpha\nu}$$

and the

$$\text{Scalar curvature:} \qquad R := R_\mu{}^\mu : g^{\mu\nu} R_{\mu\nu}$$

and the

$$\text{Einstein curvature tensor:} \qquad G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

We have found that the two proposed laws are consistent only if $R_{\rho\mu} = 0$, and not in general.

Thus the equivalence principle shouldn't be used with too much blind faith, as it never answers 2nd degree derivative questions. We should not be surprised at its failure here, because we have had to use second derivatives, which will not be the same even in the local inertial frame as they are in flat space.

The correct rule is, of course, the first, which is $D_\nu F^{\mu\nu} j = J^\mu$. The second can be ruled out because it is not covariant under *electromagnetic* gauge transformations.

Bianchi identities involving Ricci and Scalar curvatures:

$$\begin{aligned} B: &\qquad R^\alpha{}_{\beta\mu\nu;\rho} + R^\alpha{}_{\beta\nu\rho;\mu} + R^\alpha{}_{\beta\rho\mu;\nu} = 0 \\ \delta^\mu_\alpha B: &\qquad R_{\beta\nu;\rho} + R^\alpha{}_{\beta\nu\rho;\alpha} - R_{\beta\rho;\nu} = 0 \\ \frac{1}{3}\epsilon_{\kappa\lambda\alpha}{}^\beta B \epsilon^{\mu\nu\rho\lambda}: &\qquad \epsilon_{\kappa\lambda\alpha}{}^\beta R^\alpha{}_{\beta\mu\nu;\rho}\epsilon^{\mu\nu\rho\lambda} = 0 \\ &\qquad\quad = -\delta^{\mu\nu\rho}_{\kappa\alpha\beta} R^{\alpha\beta}{}_{\mu\nu;\rho} = -R^{\alpha\beta}{}_{\kappa\alpha;\beta} - R^{\alpha\beta}{}_{\alpha\beta;\kappa} - R^{\alpha\beta}{}_{\beta\kappa;\alpha} \\ &\qquad\quad = 2R^\beta{}_{\kappa;\beta} - R_{;\kappa} = 0 \end{aligned}$$

Define ${}^\sim\!G$ to be the double dual of R, that is,

$${}^\sim\!G^{\alpha\beta}{}_{\mu\nu} = \frac{1}{4}\epsilon^{\alpha\beta\gamma\delta} R_{\gamma\delta}{}^{\rho\sigma}\epsilon_{\rho\sigma\mu\nu}.$$

Define "Einstein"

$$\begin{aligned} G^\beta{}_\nu &:= {}^\sim\!G^{\alpha\beta}{}_{\alpha\nu} = -\frac{1}{4}\delta^{\beta\gamma\delta}_{\rho\sigma\nu} R_{\gamma\delta}{}^{\rho\sigma} = -\frac{1}{2}R_{\gamma\nu}{}^{\beta\nu} - \frac{1}{2}R_{\gamma\rho}{}^{\rho\beta} = R_{\rho\sigma}{}^{\rho\sigma} \\ &= R_\nu{}^\beta - \frac{1}{2}R\delta^\beta_\nu, \end{aligned}$$

so $G_{\mu\nu} = R_{\mu\nu} - \dfrac{1}{2}g_{\mu\nu}R$, and the last Bianchi identity is $G^\mu{}_{\nu;\mu} = 0$.

Note $G^\mu{}_\mu = R^\mu{}_\mu - \frac{1}{2}\delta^\mu_\mu R = -R$.

0.12 Equations Determining Geometry

Mass is the source of gravity in Newtonian mechanics. Matter must affect the metric in some way in general relativity. Let us return to the Sun.

$$\nabla^2\phi = 4\pi G\rho, \quad g_{00} = -1 - 2\phi, \quad g_{\mu\nu,0} = 0, \qquad \text{all } \Gamma \propto \phi,$$

so to first order in ϕ, $R^\alpha{}_{\beta\mu\nu} = 2\Gamma^\alpha{}_{\beta[\nu,\mu]}$, and $R^\alpha{}_{0\mu 0} = \Gamma^\alpha{}_{00,\mu}$. Thus $R_{00} = \Gamma^i{}_{00,i} - \underbrace{\Gamma^0{}_{00,0}}_{0} = -\frac{1}{2}\nabla^2 g_{00} = \nabla^2\phi = 4\pi G\rho$.

ρ is the mass density, or energy density, or T_{00}, which suggests a connection between $R_{\mu\nu}$ and $T_{\mu\nu}$. We have seen that $D_\mu T^{\mu\nu} = 0$ is an equation of motion, at least for particles in an electromagnetic field. So any connection with $T_{\mu\nu} \propto R_{\mu\nu}$ cannot be right, because $D_\mu R^{\mu\nu} \neq 0$. But $D^\mu G_{\mu\nu} \equiv 0$, so perhaps $T_{\mu\nu} \propto G_{\mu\nu}$. But for a point particle at rest, $T_{\mu\nu} = 0$ unless $\mu = \nu = 0$. So $R = -G^\mu{}_\mu = -G^0{}_0$, so $G_{00} = R_{00} - \frac{1}{2}g_{00}R = R_{00} + \frac{1}{2}G_{00}$, or $G_{00} \approx 2R_{00} \approx 8\pi G\,T_{00}$. [Note: This G is Newton's gravitational constant, not $G_\mu{}^\mu$.]

Thus we are led to guess Einstein's equation:

$$G_{\mu\nu} = 8\pi G\,T_{\mu\nu}.$$

This ia a relation between two tensors, so is covariant. Of course there is another tensor whose divergence vanishes, $g_{\mu\nu}$, and we might have written

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Λ is called the cosmological constant. It must be small because empty space (no matter) could not be flat, $G_{\mu\nu} - -\Lambda g_{\mu\nu}$ in empty space, and we therefore have had a limit $|\Lambda| < 10^{-56}\text{cm}^{-2}$, or $|\Lambda|^{-1/2} > 10^{10}$ lightyears. For calculations on motions $\ll 10^{10}$ lightyears, $|\Lambda|^{-1/2}$ might as well be ∞, $\Lambda = 0$.

When it comes to cosmology it matters whether $\Lambda = 0$ or not. Einstein originally did not include this term, but he found he could not find a stable configuration for the universe. So he postulated the existence of the cosmological term to make it possible for the universe to sit still. Of course, Hubble found that it wasn't sitting still at all, but blowing up since the big bang. Einstein called his postulating the cosmological constant "the biggest blunder of my life. But it wasn't a mistake — it may be the "dark energy" that has everyone so excited now.

0.13 Deriving the Gravitational Field Equations

Physics begins with an action:

$$S = \int d^4x\sqrt{g}\,\mathcal{L}.$$

$\mathcal{L}$ is a scalar Lagrangian density, a function of $g_{\mu\nu}$ and the matter degrees of freedom, that is, all other dynamical variables other than space-time. Divide it into

$$\mathcal{L} = \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{matter}}.$$

Here $\mathcal{L}_{\text{grav}}$ depends only on $g_{\mu\nu}$ and its derivatives, while $\mathcal{L}_{\text{matter}}$ is specified by extrapolation, using the equivalence principle, from a world where $R = 0$. So we expect $\mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{matter}}(g_{\mu\nu}, \{\psi\})$ to not depend on derivatives of g.

What scalar can we take for $\mathcal{L}_{\text{grav}}$? The only one involving two derivatives of g is R. One could also add a constant Λ.

Euler's equations need to be reconsidered as R involves second derivatives, as well as the first derivatives squared that we are used to seeing in ordinary lagrangian mechanics. We take

$$\mathcal{L}_{\text{grav}} = \frac{1}{16\pi G}R - \frac{1}{8\pi G}\Lambda$$

and vary S with respect to $g_{\mu\nu}$.

The variational pieces are[10]

$$\begin{aligned}
\delta\sqrt{g} &= \frac{1}{2}g^{-1/2}\delta(-\det g_{..}) = -\frac{1}{2}g^{-1/2}\delta e^{\text{Tr}\ln g_{..}} = +\frac{1}{2}g^{+1/2}\delta\,\text{Tr}\ln g_{..} \\
&= +\frac{1}{2}g^{+1/2}\,\text{Tr}((g_{..})^{-1}\delta g_{..} = \frac{1}{2}g^{+1/2}g^{\mu\nu}\delta g^{\mu\nu}. \\
\delta g^{\mu\nu} &= g^{\mu\nu}g_{\rho\sigma}\delta g^{\sigma\nu} = g^{\mu\nu}\left(\underbrace{\delta\left(g_{\rho\sigma}g^{\sigma\nu}\right)}_{=0} - \left(\delta(g_{\rho\sigma})\,g^{\sigma\nu}\right)\right) = -g^{\mu\rho}g^{\sigma\nu}\delta(g_{\rho\sigma})
\end{aligned}$$

[10]If the linear algebra used in $\delta\sqrt{g}$ is foreign to you, here is an alternate derivation, though you should learn the linear algebra. From "Properties of Determinants", we have $\det g_{..} = [\mu\nu\rho\sigma]\,g_{\mu\alpha}\,g_{\nu\beta}\,g_{\rho\gamma}\,g_{\sigma\delta}\,[\alpha\beta\gamma\delta]/4!$, so if we vary $g_{..}$, $\delta\det g_{..} = [\mu\nu\rho\sigma]\,g_{\nu\beta}\,g_{\rho\gamma}\,g_{\sigma\delta}\,[\alpha\beta\gamma\delta]\,\delta g_{\mu\alpha}/3! = h^{\alpha\mu}\,\delta g_{\mu\alpha}/3!$, where $h^{\alpha\mu} = [\mu\nu\rho\sigma]\,g_{\nu\beta}\,g_{\rho\gamma}\,g_{\sigma\delta}\,[\alpha\beta\gamma\delta]/3!$, so $h^{\alpha\mu}g_{\mu\phi} = 4\det g_{..}\delta^\alpha_\phi$, or $h^{\alpha\mu} = 4\det g\,g^{\alpha\mu}$. Thus $\delta\det g_{..} = 4\det g_{..}\,g^{\alpha\mu}\delta g_{\mu\alpha}$. The rest follows as on the next line.

$$\delta\Gamma^\rho{}_{\mu\nu} = +\frac{1}{2}\delta g^{\rho\sigma}\left(g_{\mu\sigma,\nu}+g_{\nu\sigma,\mu}-g_{\mu\nu,\sigma}\right)+\frac{1}{2}g^{\rho\sigma}\left(\delta g_{\mu\sigma,\nu}+\delta g_{\nu\sigma,\mu}-\delta g_{\mu\nu,\sigma}\right)$$

The first three of the six terms come to

$$\frac{1}{2}g^{\rho\alpha}\left(\delta g_{\alpha\beta}\right)g^{\beta\sigma}\left(g_{\mu\sigma,\nu}+g_{\nu\sigma,\mu}-g_{\mu\nu,\sigma}\right) = -g^{\rho\alpha}\delta g_{\alpha\beta}\Gamma^\beta{}_{\mu\nu},$$

$$\text{so } 2g_{\alpha\rho}\delta\Gamma^\rho{}_{\mu\nu} = -2\delta g_{\alpha\beta}\Gamma^\beta{}_{\mu\nu}+\delta g_{\mu\alpha,\nu}+\delta g_{\nu\alpha,\mu}-\delta g_{\mu\nu,\alpha}.$$

$$\begin{aligned}
\text{Now }\left(\delta g_{\mu\alpha}\right)_{;\nu} &= \left(\delta g_{\mu\alpha}\right)_{,\nu}-\Gamma\beta\mu\nu\delta g_{\beta\alpha}-\Gamma\beta\alpha\nu\delta g_{\mu\beta}\\
\left(\delta g_{\nu\alpha}\right)_{;\mu} &= \left(\delta g_{\nu\alpha}\right)_{,\mu}-\Gamma\beta\mu\nu\delta g_{\beta\alpha}-\Gamma\beta\alpha\mu\delta g_{\nu\beta}\\
-\left(\delta g_{\mu\nu}\right)_{;\alpha} &= -\left(\delta g_{\mu\nu}\right)_{,\alpha}+\Gamma\beta\alpha\nu\delta g_{\mu\beta}-\Gamma\beta\alpha\mu\delta g_{\nu\beta}
\end{aligned}$$

so $2g_{\alpha\rho}\delta\Gamma^\rho{}_{\mu\nu} = \delta\left(g_{\mu\alpha}\right)_{;\nu}+\delta\left(g_{\nu\alpha}\right)_{;\mu}-\delta\left(g_{\mu\nu}\right)_{;\alpha}$ and is a tensor!
and $\delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left[\delta\left(g_{\mu\alpha}\right)_{;\nu}+\delta\left(g_{\nu\alpha}\right)_{;\mu}-\delta\left(g_{\mu\nu}\right)_{;\alpha}\right]$.

Now for R:

$$\delta R^\alpha{}_{\beta\mu\nu} = \delta\left[D_\mu, D_\nu\right]^\alpha{}_\beta = \left[D_\mu,\delta D_\nu\right]^\alpha{}_\beta+\left[\delta D_\mu, D_\nu\right]^\alpha{}_\beta,$$

but $\delta D_\nu = \delta\left(\partial_\nu+\mathcal{A}_\nu\right) = \delta\mathcal{A}_\nu$, so

$$\begin{aligned}
\left[D_\mu,\delta D_\nu\right]^\alpha{}_\beta &= \delta\mathcal{A}_{\nu,\mu}+\left[\mathcal{A}_\mu,\delta\mathcal{A}_\nu\right]\\
&= \left(\delta\Gamma^\alpha{}_{\beta\nu}\right)_{,\mu}+\Gamma^\alpha{}_{\rho\mu}\delta\Gamma^\rho{}_{\beta\nu}-\Gamma^\alpha{}_{\rho\nu}\delta\Gamma^\rho{}_{\beta\mu}\\
&= \left(\delta\Gamma^\alpha{}_{\beta\nu}\right)_{;\mu}+\Gamma^\rho{}_{\nu\mu}\Gamma^\alpha{}_{\beta\rho}.
\end{aligned}$$

Subtracting $\mu\leftrightarrow\nu$ gives

$$\begin{aligned}
\delta R^\alpha{}_{\beta\mu\nu} &= \left(\delta\Gamma^\alpha{}_{\beta\nu}\right)_{;\mu}-\left(\delta\Gamma^\alpha{}_{\beta\mu}\right)_{;\nu}\\
\delta R_{\beta\nu} &= \left(\delta\Gamma^\rho{}_{\beta\nu}\right)_{;\rho}-\left(\delta\Gamma^\alpha{}_{\beta\alpha}\right)_{;\nu} \qquad \text{Palatini Identity}
\end{aligned}$$

$$\begin{aligned}
\text{And }\delta R &= \left(\delta g^{\mu\nu}\right)R_{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu}\\
&= -g^{\mu\alpha}\left(\delta g_{\alpha\beta}\right)g^{\beta\nu}R_{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu}\\
&= -\left(\delta g_{\alpha\beta}\right)R_{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu}.
\end{aligned}$$

Now $g^{\mu\nu}\delta R_{\mu\nu} = \left(g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu}\right)_{;\rho}-\left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\alpha}\right)_{;\nu}$. Each term is a divergence, $\left(A^\lambda\right)_{;\lambda} = g^{-1/2}\partial_\lambda\left(g^{1/2}A^\lambda\right)$ so $g^{1/2}g^{\mu\nu}\delta R_{\mu\nu} = \partial_\lambda(\text{something} \propto \delta\Gamma)$.

We are now ready to do the variation.

$$\begin{aligned}
0 &= \delta S = \int d^4x\,\delta\left[\sqrt{g}\left(\frac{R}{16\pi G}-\frac{1}{8\pi G}\Lambda+\mathcal{L}_{\text{matter}}\right)\right]\\
&= \int d^4x\frac{1}{2}\sqrt{g}g^{\mu\nu}\left[\frac{R}{16\pi G}-\frac{1}{8\pi G}\Lambda+\mathcal{L}_{\text{matter}}\right]\delta g_{\mu\nu}-\sqrt{g}\frac{R^{\mu\nu}}{16\pi G}\delta g_{\mu\nu}\\
&\qquad +\text{total divergence}+\sqrt{g}\frac{\delta\mathcal{L}_{\text{matter}}}{\delta g_{\mu\nu}}\delta g_{mu\nu}
\end{aligned}$$

which gives

$$\frac{-R^{\mu\nu}+\frac{1}{2}Rg^{\mu\nu}}{16\pi G}-\frac{1}{2}\frac{\Lambda}{8\pi G}+\frac{1}{2}g^{\mu\nu}\mathcal{L}_{\text{matter}}+\frac{\delta\mathcal{L}_{\text{matter}}}{\delta g_{\mu\nu}} = 0$$

or

$$G^{\mu\nu}+\Lambda g^{\mu\nu} = 16\pi G\left[\frac{\delta\mathcal{L}_{\text{matter}}}{\delta g_{\mu\nu}}+\frac{1}{2}g^{\mu\nu}\mathcal{L}_{\text{matter}}\right] =: 8\pi G\,T^{\mu\nu}$$

Here $T^{\mu\nu}$ is the stress-energy tensor of the matter. [See Weinberg p 360-363 for justification.]

0.14 Harmonic Coordinates

[Ref: Weinberg, section 7.4]

Einstein's equations are 10 equations for $G_{\mu\nu}$ or 10 differential equations for the 10 $g_{\mu\nu}$'s. But $G_{\mu\nu}{}^{;\mu} = 0$ is an identity, so 4 of the equations tell us nothing about g. This is because there is a set of four arbitrary gauge transformations corresponding to $x'^{\mu} = x'^{\mu}(x)$ arising from a change of chart. This permits us, without imposing any condition on the space-time manifold itself, to impose conditions on the chart
$\Gamma^\lambda = g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} = 0$ ["harmonic coordinate conditions"]
This can be done, given an arbitrary chart and Γ, but a chart change under which

$$\begin{aligned} \Gamma'^{\alpha'}{}_{\beta'\gamma'} &= \frac{\partial x'^{\alpha'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x'^{\beta'}}\frac{\partial x^\rho}{\partial x'^{\gamma'}}\Gamma^\mu{}_{\nu\rho} - \frac{\partial x^\nu}{\partial x'^{\beta'}}\frac{\partial x^\rho}{\partial x'^{\gamma'}}\frac{\partial^2 x'^{\alpha'}}{\partial x^\nu \partial x^\rho} \\ \text{so} \qquad \Gamma'^{\alpha'} &= \frac{\partial x'^{\alpha'}}{\partial x^\mu}\Gamma^\mu - g^{\nu\rho}\frac{\partial^2 x'^{\alpha'}}{\partial x^\nu \partial x^\rho} \end{aligned}$$

which can be set equal zero by solving this second order differential equation for $x'(x)$.

Note that a scalar has a D'Alembertian

$$\Box^2\phi := g^{\lambda\kappa}\phi_{;\lambda;\kappa} = g^{\lambda\kappa}\phi_{,\lambda\kappa} + \Gamma^\rho{}_{\lambda\kappa}\phi_{,\rho} = \phi_{,\lambda}{}^{\lambda} - \Gamma^\rho\phi_{,\rho}.$$

A harmonic function satisfies $\Box^2\phi = 0$. So thinking of x^μ as a function of x^ν,

$$\Box^2 x^\mu = x^\mu{}_{,\lambda}{}^{\lambda} - \Gamma^\rho x^\mu{}_{,\rho} = -\Gamma^\mu,$$

so x^μ is harmonic if $\Gamma^\mu = 0$.

0.14.1 The linearized Theory

[Note: from now on $\Lambda = 0$, and the following is from MTW §35, §18.1]

Suppose there exists a chart such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} \ll 1$. We work to first order in h. The quantities which are already first order, including Γ and R, can have indices raised and lowered with η. Thus

$$\begin{aligned} \Gamma^\mu{}_{\alpha\beta} &= \frac{1}{2}\eta^{\mu\nu}\left(h_{\beta\nu,\alpha} + h_{\alpha\nu,\beta} - h_{\alpha\beta,\nu}\right) \\ R^\mu{}_{\nu\rho\sigma} &= \Gamma^\mu{}_{\nu\sigma,\rho} - \Gamma^\mu{}_{\nu\rho,\sigma}, \qquad \text{so Ricci is} \end{aligned}$$

$$\begin{aligned} R_{nu\sigma} &= \Gamma^\mu{}_{\nu\sigma,\mu} - \Gamma^\mu{}_{\nu\mu,\sigma} \\ &= \frac{1}{2}\left(h_\sigma{}^\mu{}_{,\nu\mu} + h_\nu{}^\mu{}_{,\sigma\mu} - h_{\nu\sigma,}{}^\mu{}_\mu - \underbrace{h_{\nu\mu,}{}^\mu{}_\sigma}_{\text{cancels}} - h^\mu{}_{\mu,\nu\sigma} - \underbrace{h^\mu{}_{\nu,\mu\sigma}}_{\text{cancels}}\right). \end{aligned}$$

Define $h := h_\mu{}^\mu$. Given any symmetric rank 2 tensor (in particular $h_{..}$ and $R_{..}$), define $\bar{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T$, with ($T := T^\mu_\mu$). Thus

$$\begin{aligned} \bar{R}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \approx G \qquad \text{to first order} \\ \bar{\bar{T}}_{\mu\nu} &= \bar{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{T}, \quad \text{but} \quad \bar{T} = \bar{T}^\mu_\mu = T_\mu{}^\mu - \frac{1}{2}\delta^\mu_\mu T = -T \quad \text{so} \\ &= \bar{T}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}T = T_{\mu\nu}. \end{aligned}$$

$$\begin{aligned} G_{\nu\sigma} &= \frac{1}{2}(h_\sigma{}^\mu{}_{,\nu\mu} + h_\nu{}^\mu{}_{,\sigma\mu} - h_{\nu\sigma,}{}^\mu{}_\mu - h_{,\nu\sigma} \\ &\qquad -h^{\rho\mu}{}_{,\rho\mu}\eta_{\sigma\nu} + \frac{1}{2}h_{,}{}^\mu{}_\mu\eta_{\nu\sigma} + \frac{1}{2}h_{,}{}^\nu{}_\mu\eta_{\nu\sigma}) \\ &= \frac{1}{2}(\bar{h}_\sigma{}^\mu{}_{,\nu\mu} + \frac{1}{2}\eta_\sigma{}^\mu h_{,\nu\sigma} \\ &\qquad +\bar{h}_\nu{}^\mu{}_{,\sigma\mu} + \frac{1}{2}\eta_\nu{}^\mu h_{,\sigma\mu} \\ &\qquad -\bar{h}_{\nu\sigma,}{}^\mu{}_\mu - \frac{1}{2}\eta_{\nu\sigma}h_{,}{}^\mu{}_\mu - h_{,\nu\sigma} \\ &\quad -\bar{h}^{\rho\mu}{}_{,\rho\mu}\eta_{\sigma\nu} - \frac{1}{2}\eta^{\rho\mu}h_{,\rho\mu}\eta_{\sigma\nu} + h_{,}{}^\mu{}_\mu\eta_{\nu\sigma}) \\ &= \frac{1}{2}(\bar{h}_\sigma{}^\mu{}_{,\nu\mu} + \frac{1}{2}h_{,\nu\mu} \\ &\qquad +\bar{h}_\nu{}^\mu{}_{,\sigma\mu} + \frac{1}{2}h_{,\sigma\nu} \\ &\qquad -\bar{h}_{\nu\sigma,}{}^\mu{}_\mu - \frac{1}{2}\eta_{\nu\sigma}h_{,}{}^\mu{}_\mu - h_{,\nu\sigma} \\ &\quad -\bar{h}^{\rho\mu}{}_{,\rho\mu}\eta_{\sigma\nu} - \frac{1}{2}h_{,}{}^\mu{}_\mu\eta_{\sigma\nu} + h_{,}{}^\mu{}_\mu\eta_{\nu\sigma}) \\ &= \frac{1}{2}\left(\bar{h}_\sigma{}^\mu{}_{,\nu\mu} + \bar{h}_\nu{}^\mu{}_{,\sigma\mu} - \bar{h}_{\nu\sigma,}{}^\mu{}_\mu - \bar{h}_{\rho\mu}{}^{,\rho\mu}\eta_{\sigma\nu}\right) = 8\pi G\, T_{\sigma/nu} \end{aligned}$$

Next, we turn to a particular choice of chart, the harmonic coordinates

$$\Gamma^\lambda = \eta^{\alpha\beta}\Gamma^\lambda{}_{\alpha\beta} = h^\lambda{}_\alpha{}^{,\alpha} - \frac{1}{2}h^{,\lambda} = \bar{h}^\lambda{}_\alpha{}^{,\alpha} = 0.$$

This is harmonic gauge, also called lorentz gauge. In this gauge,

$$G_{\nu\sigma} = -\frac{1}{2}\bar{h}_{\nu\sigma,\ \mu}^{\ \ \ \ \mu} - 8\pi G\, T_{\nu\sigma}.$$

Suppose $T_{\mu\nu} = 0$ except $T_{00} = M\delta^3(\vec{r})$, a point source. Then $\bar{h}_{00} = 4MG/r$, $\bar{h}_{0i} = \bar{h}_{ij} = 0$, $h_{00} = h_{xx} = h_{yy} = h_{zz} = 2MG/r$, and other components are zero. Converting to spherical polar spatial coordinates $dt, dr, d\theta, d\phi$,

$$\mathbf{g} = -\left(1 - \frac{2GM}{r}\right) dt\otimes dt + \left(1 + \frac{2GM}{r}\right)\left(dr\otimes dr + r^2 d\theta\otimes d\theta + r^2\sin^2\theta\, d\phi\otimes d\phi\right).$$

You may verify that Riemann has components given in MTW 1.14, most easily using x, y, z coordinates at a point $x = y = 0, z = r$.

The solution we just found is a solution, but it is not unique. For the homogeneous equation $G_{\nu\sigma} = -\frac{1}{2}\Box^2\bar{h}_{\nu\sigma} = 0$ for gravity in "empty" space has nontrivial plane wave solutions $\bar{h}^{\mu\nu}(x^\rho) = \bar{h}^{\mu\nu}(z-t)$, a plane wave which we have taken, arbitrarily, in the z direction.

Consider a gauge change $x'^{\mu} = x^\mu + \xi^\mu(x)$.

$$\begin{aligned} g'_{\mu\nu} &= \eta_{\mu\nu} + h'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^{\mu}}\frac{\partial x^\sigma}{\partial x'^{\nu}}(\eta_{\rho\sigma} + h_{\rho\sigma}) \\ &= \left(\delta^\rho_\mu + \xi^\rho_{\ ,\mu}\right)\left(\delta^\sigma_\nu + \xi^\sigma_{\ ,\nu}\right)(\eta_{\rho\sigma} + h_{\rho\sigma}) \\ &= \eta_{\mu\nu} + h_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu}, \end{aligned}$$

so

$$\begin{aligned} h'_{\mu\nu} &= h_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu}, \qquad h' = h - 2\xi_\mu^{\ ,\mu} \\ \bar{h}'_{\mu,\nu} &= \bar{h}_{\mu,\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu} + \xi_\mu^{\ ,\mu}\eta_{\mu,\nu}. \end{aligned}$$

This is the same kind of transformation we used to make h' harmonic.

$$\bar{h}'_{\mu\nu}{}^{,\nu} = \bar{h}_{\mu\nu}{}^{,\nu} - \underbrace{\xi_\nu{}^{,\nu}{}_\mu}_{\text{cancels}} - \xi_{\mu,\nu}{}^{\nu} + \underbrace{\xi_\rho{}^{,\rho\nu}\eta_{\mu\nu}}_{\text{cancels}},$$

which could be set equal to zero by finding ξ such that $\Box^2\xi_\mu = barh^{\text{old}}{}_{\mu\nu}{}^{,\nu}$. If h^{old} is already harmonic, we can still make a gauge change providing $\Box^2\xi_\mu = 0$, and h remains harmonic. For our plane wave, choose $\xi_\mu = \xi_\mu(z-t)$, which satisfies this equation. Choose $\xi_z = \int^{z-t}\bar{h}^{\text{old}}{}_{zz}(u)$, $\xi_t = 0$, so $\xi_{z,z} = \bar{h}^{\text{old}}{}_{zz}$ and

$\bar{h}^{\text{new}}{}_{zz} = 0$. Also choose $\xi_{x,y} = \int du \bar{h}_{z,x \text{ or } y}(u)$ so $h^{\text{new}}_{xz} = 0$. Now drop the *new* notation. $0 = \bar{h}^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu z}{}_{,z} + \bar{h}^{\mu t}{}_{,t} = \bar{h}^{\mu z'} - \bar{h}^{\mu t'}$ so $\bar{h}^{\mu t} = \bar{h}^{\mu z}$ for all μ. Therefore

$$\bar{h}^{zt} = \bar{h}^{zz} = 0 = \bar{h}^{tz} = \bar{h}^{tt} = \bar{h}^{xz} = \bar{h}^{yz} = \bar{h}^{xt} = \bar{h}^{yt}.$$

This leaves only $\bar{h}^{xx}$, $\bar{h}^{xy}$ and $\bar{h}^{yy}$, so only these three components are left.

Recall that it is the Riemann curvature which is the physically relevant object. (h is not even a tensor, as it is $g - \eta$ and η isn't!) So we calculate

$$R_{\mu\nu\rho\sigma} = \Gamma_{\mu\nu\sigma,\rho} - \Gamma_{\mu\nu\rho,\sigma} = \frac{1}{2}\left(\underbrace{h_{\nu\mu,\sigma\rho}}_{\text{cancels}} + h_{\sigma\mu,\nu\rho} - h_{\nu\sigma,\mu\rho} - \underbrace{h_{\nu\mu,\rho\sigma}}_{\text{cancels}} - h_{\rho\mu,\nu\sigma} + h_{\nu\rho,\mu\sigma}\right).$$

But $\bar{h}_{\mu\nu,\rho\sigma} = 0$ unless $\mu = x$ or y, $\nu = x$ or y, $\rho = t$ or z, $\sigma = t$ or z, (and $h_{\mu\nu,zz} = -h_{\mu\nu,zt} = h_{\mu\nu,tt}$),

$$h_{\mu\nu,\rho\sigma} = \bar{h}_{\mu\nu,\rho\sigma} + \frac{1}{2}\eta_{\mu\nu}h_{,\rho\sigma}, \qquad \text{so} \quad h = -\bar{h}^x{}_x - barh^y{}_y$$

$$\begin{aligned} h_{xx,tt} &= \frac{1}{2}\left(\bar{h}_{xx} - \bar{h}_{yy}\right)_{,tt} =: 2a \\ h_{yy,tt} &= \frac{1}{2}\left(\bar{h}_{yy} - \bar{h}_{xx}\right)_{,tt} \\ h_{zz,tt} &= -\frac{1}{2}\left(\bar{h}_{xx} + \bar{h}_{yy}\right) \\ h_{tt,tt} &= +\frac{1}{2}\left(\bar{h}_{xx} + \bar{h}_{yy}\right) \\ h_{xy,tt} &= \bar{h}_{xy,tt} =: 2b \end{aligned}$$

other $h_{\mu\nu,tt} = 0$, $h_{\mu\nu,zz} = -h_{\mu\nu,zt} = h_{\mu\nu,tt}$.

Now for R's. There are cases:

- One index is x or y. By symmetry, it can always be made the first.

$$R_{x\nu\rho\sigma} = \frac{1}{2}\left(h_{x\sigma,\nu\rho} - h_{x\rho,\nu\sigma}\right).$$

  One of the other indices must be x or y. If x, it cannot be ν by antisymmetry, so take it to be ρ. Then ν and σ must be z or t, and $R_{x\nu x0} = -\frac{1}{2}h_{xx,\nu\sigma}$, so $R_{x0x0} = -a$, $R_{xzx0} = a$, $R_{xzxz} = -a$.

The other index could be y, wich still cannot be ν, so $R_{x\nu y\sigma} = -\frac{1}{2}h_{xy,\nu\sigma}$; $R_{x0y0} = -b = -R_{xzy0} = R_{xzyz} = -R_{x0yz}$.

Similarly $R_{y0y0} = a$, $R_{yzy0} = -a$, $R_{yzyz} = a$

- Finally, no index might be x or y. But then we have only

$$R_{z0z0} = \frac{1}{2}\left(\underbrace{h_{0z,0z}}_{\text{cancels}} - h_{tt,zz} - h_{zz,tt} + \underbrace{h_{0z,0z}}_{\text{cancels}}\right) = -\frac{1}{2}\left(h_{tt,tt} + h_{zz,tt}\right) = 0.$$

Consider a ruler in an inertial coordinate system. To first order in the length of the ruler, coordinates and distances agree as $g = \eta$ exactly at one end. Consider two masses, one at the origin and one at position $\vec{r}$, also in free fall and approximately at rest. Then

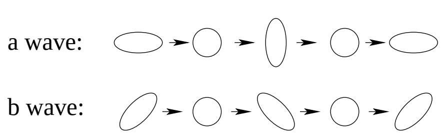
$$\frac{d^2 r^i}{dt^2} = -R^i{}_{0j0} r^j.$$

thus z is not affected, and

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

where $a = \frac{1}{4}\left(\bar{h}_{xx} - \bar{h}_{yy}\right)_{,tt}$, $b = \frac{1}{2}\bar{h}_{xy,tt}$ are the two degrees of freedom in the gravitational wave.

Consider a ring of such particles in the xy plane at a fixed z. Pure a or pure b waves $\propto e^{i\omega t}$ will deform a ring as shown.



This is a quadripole deformation and indicates a spin 2 object quantum mechanically. The gravitational wave is clearly propagating at the speed of light, so we have shown that the graviton (a quantum of gravitational wave) is a spin 2 massless particle.

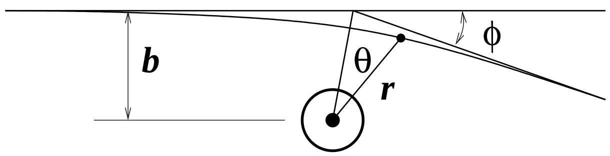
If the graviton is a spin two particle, why are there only two possible polarizations represented by a and b? For massive particles we know that, in the rest frame, we must have z-components of spin with values $(s, s-1, s-2, \ldots - s)\hbar$, which would be 5 polarizations for a spin two object. But for massless particles there is no rest frame. We can discuss helicities, that is, the component of spin in the direction of motion. But there is no longer an

angular momentum lowering operator to connect one helicity to another, and a pure spin 2 massless particle has only helicity ± 2, never ± 1 or 0. This is analogous to the polarization of photons, which have helicity ± 1, never zero. Said another way, the polarizations are always transverse, not longitudinal.

0.15 The Bending of Light

The geodesic equation for a massless particle is

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}.$$



Consider a photon initially travelling on a line with impact parameter b with respect to the center of the Sun. Then we can take $x = t$ $y = b$ initially, with $z = 0$. λ is a arbitrary path parameter, but we can choose it to be roughly $x = t$ plus some first order corrections, so

$$\frac{dx^\mu}{d\lambda} = (1,1,0,0) + \mathcal{O}(GM/b), \qquad \text{and}$$

$$\begin{aligned}
\frac{d}{dx}\frac{dy}{dx} &= -\Gamma^y{}_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = -\Gamma^y{}_{00} - \Gamma^y{}_{00} - 2\Gamma^y{}_{x0} - \Gamma^y{}_{xx} \\
&= \frac{1}{2}h_{00,y} + \frac{1}{2}h_{xx,y} \qquad \text{as} \quad h_{xy} = h_{x0} = h_{y0} = 0 \quad \text{and} \quad h_{\mu\nu,0} = 0 \\
&= \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{2MG}{r} + \frac{2MG}{r}\right) \\
&= -2MG\frac{y}{r^3} = -2MG\frac{b}{(x^2+b^2)^{3/2}},
\end{aligned}$$

$$\text{so} \quad \frac{dy}{dx} = \frac{-2MG}{b}\int \frac{d(\tan\theta)}{(1+\tan^2\theta)^{3/2}} = \frac{-2MG}{b}\int_{-\pi/2}\cos\theta d\theta$$

and $\left.\frac{dy}{dx}\right|_{\text{final}} = \frac{-2MG}{b}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = -\frac{4MG}{b} = -\phi$, where ϕ is the bending angle. So

$$\phi = \frac{4M_\odot G}{R_\odot}\frac{R_\odot}{b} = 1.75''\frac{R_\odot}{b}.$$

Of course $b > R_\odot$, the Sun's radius, so the bending of light is quite small, less than 1.75 seconds of arc.

[New topic?]

The general solution to $\bar{h}_{\nu\sigma}{}^{,\mu}{}_\mu = -16\pi G T_{\nu\sigma}$ is $\bar{h}_{\nu\sigma} = \int \frac{4GT_{\nu\sigma}(t-|\vec{x}-\vec{x}\,'|,\vec{x}}{|\vec{x}-\vec{x}\,'|}$ free waves. The proof is for general f. If $f^{,\mu}{}_\mu = -4\pi\rho(t,\vec{x})$ then

$$f = \int \frac{\rho(t-|\vec{x}-\vec{x}\,'|,\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}d^3x',$$

for then

$$f^{,\mu}{}_\mu = -\int \frac{\ddot{\rho}(t-|\vec{x}-\vec{x}\,'|,\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}d^3x' + \int \vec{\nabla}^{\,2}\left(\frac{\rho(t-r,vecpx)}{r}\right)_{r=|\vec{x}-\vec{x}\,'|}d^3x'.$$

$$\begin{aligned}
\text{But} \qquad \vec{\nabla}^{\,2}\frac{\rho}{r} &= \frac{1}{r}\underbrace{\vec{\nabla}^{\,2}\rho(r)}_{\frac{\partial^2\rho}{\partial r^2}+\frac{2}{r}\frac{\partial\rho}{\partial r}} + \underbrace{2\vec{\nabla}\frac{1}{r}\cdot\vec{\nabla}\rho}_{-\frac{2}{r^2}\frac{\partial\rho}{\partial r}} - 4\pi\rho(0)\delta^3(\vec{r}) \\
&= \frac{1}{r}\ddot{\rho} - -4\pi\rho(0)\delta^3(\vec{x}-\vec{x}\,')
\end{aligned}$$

where we are treating ρ as a function of t and r for fixed $\vec{x}\,'$, so it depends only on the magnitude of r, not its direction. So we see

$$f^{,\mu}{}_\mu = -4\pi\rho(t,\vec{x}\,').$$

Now suppose nothing about $T_{\mu\nu}$ except that at time t, it is confined to some compact region of space. Then for $\vec{x}$ far from that region, in which we place $\vec{x}\,' = 0$, we might expect that the details of the sources would be largely lost, and we would get the Newtonian expression, even if the sources themselves are moving rapidly. MTW assigns as exercise 19.1(a) and 19.3(b,c) the derivation of the form of the metric, in the source's center of mass, as

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2M}{r} + \mathcal{O}r^{-3}\right)dt^2 - 4\epsilon_{jk\ell}fracS^kx^\ell r^3 dt\otimes dx^j \\
&\quad + \left(1 + \frac{2M}{r} + \mathcal{O}r^{-1}\cdot\text{wave terms} + \mathcal{O}r^{-2}\right)dx^i\otimes dx^j,
\end{aligned}$$

where $M + \int T_{00}d^3x$, $S_k = \int \epsilon_{k\ell m}x^\ell T^{m0}d^3x$.

Note that we are working to first order in G, and hence in h, but making no restrictions on the sources (*i.e.* they needn't be nonrelativistic) except for being, at a fixed time, in a region which is limited to being much closer to the origin than $\vec{r}$.

A nonrelativistic test particle circling the source will feel only g_{00} and hence see a Newtonian $\phi = -GM/r$, where the total mass is the integral of the local mass density T_{00}. This is how one measures the mass.

If one performs a more sensitive test, however, one can measure the angular momentum of the source $\vec{S}$. We will consider a gyroscope supported to stay at a fixed coordinate. the gyro is not freely falling. Nonetheless, at each instant, we can erect a local inertial frame so that we can discuss what happens to the spin of the gyro $\vec{L}$ as it is accelerated. In the momentary rest frame, we know $d\vec{L}/dt = 0$, for there is no torque on the gyro. Thus $dS^\mu/d\tau \propto u^\mu$. Also in this frame $d\vec{u}/d\tau = \vec{a}$. From $L \cdot u = 0$, we find

$$\frac{dL}{d\tau} \cdot u + L \cdot \frac{du}{d\tau} = 0, \qquad \frac{dL}{d\tau} = -uL \cdot \frac{du}{d\tau} = a \cdot Lu.$$

We wish to express these equations in covariant form. Clearly

$$\frac{DL}{d\tau} = a \cdot Lu, \quad \text{with} \quad a := \frac{Du}{d\tau}.$$

We hold our gyro at a fixed point x, y, z, t in the coordinate system of Eq. 19.5. Then $u = \partial_t, u^\mu = (1,0,0,0)$. In our coordinate system

$$\frac{Du_\mu}{D\tau} = \frac{\partial u^\mu}{\partial \tau} + \gamma^\mu{}_{\rho\sigma} u^\rho u^\sigma = 0 + \Gamma^\mu{}_{00} = a^\mu,$$

$$\Gamma^\mu{}_{00} = -\frac{1}{2} g_{00,}{}^\mu = \left(0, \frac{m}{r^3} x^i\right)$$

[Note it is accelerating outward]

$$\frac{DL^\mu}{d\tau} = aLu^\mu = \gamma^\mu{}_{\rho 0} L^\rho + \frac{dL^\mu}{d\tau}$$

$$\Gamma_\mu{}^{i0} = -\frac{1}{2} g_{i0,}{}^\mu + \frac{1}{2} g^\mu{}_{0,i} \Longrightarrow \Gamma^0{}_{i0} = +a^i, \quad \Gamma^j{}_{i0} = \left(-\epsilon_{jk\ell} \frac{S^k x^\ell}{r^3}\right)_{,i} - (i \leftrightarrow j)$$

so $\dfrac{dL^0}{d\tau} = a \cdot L - a \cdot L = 0, \qquad \dfrac{dL^i}{d\tau} = -\gamma^i{}_{j0} L^j = \epsilon_{inr} \Omega^n L^r$, where $\Omega^m = \frac{1}{2} \epsilon_{nk\ell} \Gamma^k{}_{\ell 0}$, $\epsilon_{k\ell n} \Omega^n = \Gamma^k{}_{\ell 0}$.

Ω is a vector which describes the rotation rate of L,

$$\Omega^n = \epsilon_{nk\ell} \left[-\epsilon_{kjm} \left(\frac{S^j x^m}{r^3} \right)_{,\ell} \right] = \epsilon_{kn\ell} \epsilon_{kjm} S^j \left(\frac{\delta^m_\ell}{r^3} - 3 \frac{x^m x^\ell}{r^5} \right) = -\frac{S^n}{r^3} + 3 \frac{x \cdot S x^n}{r^5}.$$

Consider a spherically symmetric, static star. Then $g_{\mu\nu}$ can be chose to be a function of r, and rotationally invariant. One can show that by redefinition of coordinates one can choose

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 \underbrace{d\Omega^2}_{(d\theta)^2 + \sin^2\theta d\phi^2}$$

[or φ ?]
where $\Phi(r)$ and $\Lambda(r)$ are two unknown (so far) functions. The choice of coordinates is called **Schwarzschild coordinates**. In particular, r is $\frac{1}{2\pi}$ times the proper circumference of a circle at fixed r, $\theta = \pi/2$, with $\phi \in [0, 2\pi)$, or it is the square root of $1/4\pi$ times the area of a sphere at constant r.

Homework due Nov 14:

1. Exercise 23.4 — Find $G_{\mu\nu}$ for the Schwarzschild coordinates 23.7 [Answers on page 360]

2. Show that outside the matter distribution, where $T_{\mu\nu} = 0$, $\Phi' = -\Lambda'$, and if we require $\Phi \underset{r\to\infty}{\longrightarrow} 0$, $\Lambda \underset{r\to\infty}{\longrightarrow} 0$, then $\Phi = \ln\sqrt{1 - \dfrac{K}{r}} = -\Lambda$, so comparing at large r with the weak field solution, $K = 2M$, and Eq. 23.27:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

   This is known as the Schwarzschild metric.

This is the metric on the outside of the distribution of matter. We will return to a consideration of this after treating the distribution itself.

The actual physical applications are, of course, stars, and we therefore can put off no longer a discussion of $T_{\mu\nu}$ for an ideal gas.

Consider a perfect fluid, which means that a small element of fluid has an average rest frame, in which the surrounding fluid is isotropic, suppoprting no shear or heat flow. In the rest frame, the energy density is ρ by definition [actually in the local inertial frame with respect to which the fluid is momentarily at rest]. So $T^{00} = \rho$. There is no momentum density because we are in teh rest frame, so $T^{j0} = 0$. But there is a momentum flux. Consider a small area in space $d\vec{S}$, The material on one side exerts a force on the other side equal to $pd\vec{S}$, where p is the pressure. As we forbid shear forces, the force is normal to the surface, $\vec{F} = pd\vec{S}$, and momentum is being transferred across the surface at a rate $\frac{dP^i}{dt} = pdS^i$. But this is just the defining exapression for T_{ij}, so $T^{ij} = p\delta_{ij}$ in the local inertial rest frame. Thus

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} = \rho u^\mu u^\nu + p\left(g^{\mu\nu} + u^\mu u^\nu\right)$$

in the local inertial frame. I claim that ρ and p, being defined in any frame in terms of the local inertial rest frame, are scalars. Then

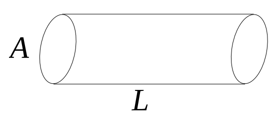
$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu}$$

is a tensor equation and must be true in any frame. [But with u the mean 4-velocity of the gas element]

This result has some funny properties quite independent of general relativity. The energy density in an inertial frame in flat space, in which the fluid is moving with the velocity $\vec{v}$ is (as $u = (\gamma, \gamma\vec{v})$)

$$T^{00} = (\rho + p)\gamma^2 - p = \rho\gamma^2 + pv^2\gamma^2.$$

The first term is just what you might expect. The energy in a container, say a cylinder with its axis along the direction of motion, is, in the rest frame,



$$\left.\begin{array}{c} E = T^{00}V = \rho A L \\ p = 0 \\ \text{In the other coordinate system we might expect} \\ E' = E\gamma, \quad L' = L/\gamma \quad \text{(length contraction)} \\ A' = A \quad (\perp \text{ lengths unchanged}) \\ \text{so} \quad T^{00\,\prime} = E'/V' = T^{00}\gamma^2 = \rho\gamma^2, \end{array}\right\} \quad \text{Not right!}$$

and we don't see why the pressure should enter at all. What is the origin of the second term?

One's first thought is that the pressure term is natural because if you compress a gas, you do work on it, and increase its energy. But that is already in ρ, which is the energy density (in the rest frame) and includes the kinetic energy of random motion and the potential energy dues to intermolecular forces. In fact, work done compressing a gas increased the energy in the rest frame (where one normally discusses this issue nonrelativistically) where the second term vanishes, as $\vec{v} = 0$.

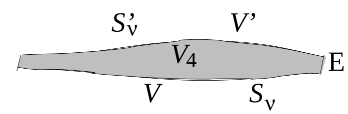
The origin of our paradoc lies in what we mean by the energy of a gas. In fact, our genral impression that the energy-momentum vector is a vector needs to be better defined.

Suppose we have a system with an energy density $\epsilon(\vec{x}, t)$. We define the energy of the system to be

$$E(t) = \int \epsilon(\vec{x}, t) dV \qquad \text{at a fixed time.}$$

Another observer calculates $E'(t') = \int \epsilon'(\vec{x}\,', t')dV'$, integrating over a completely different slice of spacetime.

In the case of an isolated system, this problem of different hypersurfaces is not crucial. In general one observer calculates $P^\mu = \int_V T^{\mu\nu} dS_\nu$, while the other calculates



$$\begin{aligned} P'^{\,\mu} &= \int_{V'} T'^{\,\mu\nu} dS'_\nu \\ &= \int_{V'} \frac{\partial x'^{\,\mu}}{\partial x^\rho} T^{\rho\nu} dS_\nu \\ &= \Lambda^\mu{}_\rho \underbrace{\int_{V'} T^{\rho\nu} dS_\nu}_{\int_{V_4} T^{\rho\nu}{}_{,\nu} d^4x + \int_V T^{\rho\nu} dS_\nu + \int_E T^{\rho/nu} dS_\nu} \end{aligned}$$

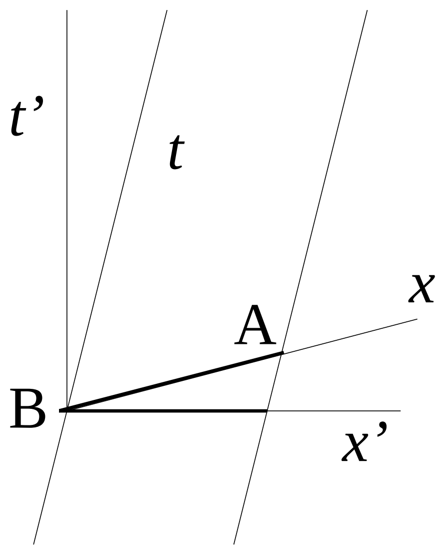
Here V and V' are the volume integrals for the two observers, E is a 3-dimensional hypersurface connecting V and V', and V_4 is the hypervolume enclosed. Now **<u>if</u>** $T^{\mu\nu}$ is conserved and **<u>if</u>** it is localized to a finite region of spacetime not including the "edges" E, we have $P'^\mu = \Lambda^\mu{}_\rho P^\nu$ as expected. But for a fluid element under pressure this isolation is not possible. Either we are cutting off the volume integral where there is still $T_{\mu\nu}$, or if by $T_{\mu\nu}$ we include only that of the gas and not the container, then $T_{\mu\nu}$ is not conserved because it leaves the gas at the walls.

How much energy is contained in the fluid in the cylinder? Arrange to pop the lids of the cylinder simultaneously, so the gas spills out without any work being done. Afterwards the gas is isolated and the energy of the gas is just what it was when it was contained.

Now if observer $C = x$, riding with the cylinder, pops the lids off simultaneously, the final escaped gas will have an energy (in his reference frame) of $\int T^{00}{}_C dV_C = \rho A L_C$, and total momentum $\vec{P} = 0$. This isolated gas, observed by the x' = laboratory observer $\mathcal{O}_{\rm lab}$ must therefore have energy

$$E' = \gamma E_C + \gamma v P_c = \gamma E_C = \gamma \rho A L,$$

but this is not the energy which was in the gas to begin with, because, according to $\mathcal{O}_{\rm lab}$, the front lid at A popped at time $t' = \gamma^2 vŁ'$ later than the back lid at B. During that time the fluid was exerting a force on the front lid, which in the cylinder's frame had components $(F^\mu = (0, pA)$, giving rise to a net transfer of momentum $(0, pA\Delta t) = (0, pAL/v)$ from the gas to the walls. This momentum has been included in what $\mathcal{O}_{\rm lab}$ considered the fluid's 4-momentum before the pop.



$A = (t = 0, x = L) = (t' = t'_A, x'_A)$
$t' = \gamma t + \gamma v x$, so $t'_a = \gamma v L = \gamma^2 v L'$

So converting to his frame $E' = \gamma E + \gamma v P = \gamma v p A L v$ So the total energy in the fluid before the lid popped was

$$E' = AL(\gamma\rho + \gamma v^2 p),$$

and the volume $V' = AL' = AL/gamma$, so

$$T'^{00} = E'/V' = \gamma^2\rho + \gamma^2 v^2 p$$

as required by the tensor argument

0.16 Perfect Fluids

Consider a small volume element of a perfect fluid, with

$$\begin{aligned}
\rho &= \text{energy density in the local inertial rest frame}\\
p &= \text{pressure in the local inertial rest frame}\\
n &= \text{baryon number density in the local inertial rest frame}\\
u^\mu &= \text{four-velocity of the element}\\
&\quad (= (1,0,0,0) \text{ in the local inertial rest frame}
\end{aligned}$$

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu}.$$

If a star is to be static in the Schwarzschild coordinates, $r^r = u^\theta = u^\phi = 0$, and $g_{tt}(u^t)^2 = -1$, so let $u^t = e^{-\Phi(r)}$, $T^{00} = \rho e^{-2\Phi}$, $T^{rr} = p e^{-2\Lambda(r)}$, $T^{\theta\theta} = \dfrac{p}{r^2}$, $T^{\phi\phi} = \dfrac{p}{r^2 \sin^2\theta}$,

$$\begin{aligned}
T^{\mu\nu}{}_{;\nu} &= (\rho+p)_{,\nu} u^\mu u^\nu + (\rho+p) u^\mu{}_{,\nu} u^\nu + (\rho+p)\Gamma^\mu{}_{\rho\nu} u^\rho u^\nu\\
&\qquad + (\rho+p) u^\mu \underbrace{u^\nu{}_{;\nu}}_{\underbrace{u^\nu{}_{,\nu}}_{0} + \Gamma^\nu{}_{\rho\nu} u^\rho} + p_{,\nu} g^{\mu\nu}\\
&= (\rho+p)_{,0} (u^0)^2 \delta^\mu_0 + (\rho+p) \underbrace{u^\mu{}_{,0}}_{0} u^0\\
&\qquad + (\rho+p)\left(\Gamma^\mu{}_{00}(u^0)^2 + \underbrace{\Gamma^\nu{}_{0\nu}}_{g^{-1/2}\partial_0 g^{1/2} = 0} u^0 u^\mu\right) + p_{,r} g^{\mu r}\\
&= (\rho+p) \underbrace{\Gamma^\mu{}_{00}}_{\delta^\mu_r \Phi' e^{2(\Phi-\lambda)}} (u^0)^2 + p_{,r} g^{\mu r}\\
&= 0,
\end{aligned}$$

so we only get an equation fro, $\mu = r$,

$$(\rho + p)\Phi' + p_{,r} = 0.$$

The equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ will determine Φ and Λ in terms of ρ and p. The equation $T^{\mu\nu}{}_{;\nu} = 0$ gives differential equations for ρ and p analogous to those one findes in Newtonian physics, but not enough to find ρ

and p. We also need an equation of state for the fluid, in the form, perhaps, of $p = p(\rho, T)$, $\rho = \rho(p, T)$ or similar equations in terms of entropy, $p(n, S)$. In practical applications there is some approximation which permits a simplification. For example, white dwarfs and neutron stars are essentially degenerate fermi gases, with temperature far below the fermi temperature. Therefore p and ρ are affectively independent of T and are functions solely of n, which determines the fermion density.

Let us return to Einstein's equations. in terms of the orthonormal basis

$$\mathbf{e}_{\hat t} = e^{-\Phi}\partial_t, \qquad \mathbf{e}_{\hat r} = e^{-\Lambda}\partial_r, \qquad \mathbf{e}_{\hat\theta} = \frac{1}{r}\partial_\theta, \qquad \mathbf{e}_{\hat\phi} = \frac{1}{r\sin\theta}\partial_\phi,$$

you should find

$$\begin{aligned} G_{\hat t\hat t} &= \frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) + \frac{2}{r}e^{-2\Lambda}\Lambda' \\ &= \frac{1}{r^2}\frac{d}{dr}\underbrace{\left\{r\left(1 - e^{-2\Lambda}\right)\right\}}_{\text{call this } 2m(r)} = 8\pi T_{\hat t\hat t} = 8\pi\rho, \end{aligned}$$

so $m(r) = \int_0^r 4\pi r^2\rho(r)dr$, where we require $m(0) = 0$ so g won't be singular there. $m(r)$ is the mass within the radius r. While this might seem obvious because $\rho(r)$ is the mass density, it isn't obvious. $\rho(r)$ is the mass per unit proper volume. The distance between r and $r+dr$ is $e^\Lambda dr$, not dr, so m is not the sum of rest mass energies. The rest mass energy in the shell is therefore $4\pi r^2\rho(r)(1-2m/r)^{-1/2}dr$ so the extra term $-4\pi r^2\rho(r)[-1+(1-2m/r)^{-1/2}]dr$ is the gravitational potential energy, and $\underset{G\to 0}{\longrightarrow} 4\pi r^2\rho\frac{GM}{r}$.

Let's review:

$$\begin{aligned} ds^2 &= -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2 \\ T^{\mu\nu} &= (\rho + p)u^\mu u^\nu + p g^{\mu\nu} \\ G^{mu\nu} &= 8\pi G\, T^{\mu\nu} \end{aligned}$$

From ${T^{\mu\nu}}_{;\nu} = 0$, $\quad (\rho + p)\Phi' + p_{,r} = 0. \qquad$ Define $m(r) = \int_0^r 4\pi r'^2\rho(r')dr'$.

From $G_{\hat t\hat t} = 8\pi G\, T_{\hat t\hat t}$, $\qquad \dfrac{2Gm(r)}{r} = 1 - e^{-2\Lambda}$

From your homework:

$$G_{\hat r\hat r} = \underbrace{-\frac{1}{r^2}\left(1 - e^{-2\Lambda}\right)}_{-\frac{2m(r)}{r^3}} + \frac{2}{r}\underbrace{e^{-2\Lambda}}_{\left(1-\frac{2m}{r}\right)}\Phi' = 8\pi T_{\hat r\hat r} = 8\pi p$$

$$\begin{aligned} \Phi' = \frac{4\pi p r + \frac{m}{r^2}}{1 - \frac{2m}{r}} &= \frac{4\pi r^3 p + m}{r(r-2m)} \\ &= -\frac{1}{\rho + p}\frac{dp}{dr}. \end{aligned}$$

The last equality is known as the **Oppenheimer-Volkov Equation**.

Compared to the nonrelativistic form $\dfrac{dp}{dr} = -\dfrac{\rho m}{r^2}$ the Oppenheimer-Volkov equation

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}$$

requires a larger dp/dr at the same r. Thus the pressure must rise faster as one penetrates the star.

We now have a set of coupled ordinary differential equations for m (equivalently Λ), p, and ρ: $\quad \dfrac{dm}{dr} = 4\pi r^2\rho(r), \quad \dfrac{dp}{dr} =$ Oppenheimer-Volkov, and the equation of state $\rho(p)$. These can be solved numerically given the equation of state. The procedure is to start at the origin. Choose a $\rho(0)$, integrate out until you get to zero pressure, which defines the surface of the star. You don't know in advance what mass star you will get, so you adjust $\rho(0)$, and try again, or run a series of values of $\rho(0)$.

Sometimes it is nice to have an analytic solution to think about. Artificial equations of state may have analytic solutions. If $\rho(p) = \rho_0\Theta(p)$, a star of constant density, then clearly $M =$ mass of star $= \dfrac{4\pi}{3}\rho_0 R^3$. The solution gives

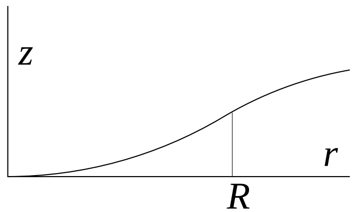
$$p_c = \rho_0\frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1} \qquad = \text{central pressure.}$$

Note if $R = R_{\text{Lim}} = (3\pi\rho_0)^{-1/2}$, $M_{\text{Lim}} = \dfrac{4}{9}R_{\text{Lim}}$, $p_{c\,\text{Lim}} = \infty$ is the limit to how much mass one can stabily insert.

The solution for the metric gives trivially

$$e^\Lambda = \left(1 - \frac{2m(r)}{r}\right)^{-1/2} = \left(1 - \frac{8\pi}{3}\rho_0 r^2\right)^{-1/2} = \frac{ds}{dr},$$

where $s(r)$ is the proper distance to the origin. This geometry may be pictured if we forget about time and one spatial dimension, as a two dimensional curved surface which, in our imagination, we may enbed in three dimensions. The z dimension is physically meaningless, used only as room for the two dimensional world to curve. As $(ds)^2 = dz^2 + dr^2 = \dfrac{(dr)^2}{1 - \frac{8\pi}{3}\rho_0 r^2}$ for fixed angle, $z = \sqrt{\dfrac{3}{8\pi\rho_0}}\sqrt{1 - \dfrac{8\pi}{3}\rho_0 r^2}$, which is a piece of a sphere, of fixed radius. The spherical arc extends out to the radius of the star.



Thereafter,

$$e^\Lambda = \left(1 - \frac{2M}{r}\right)^{-1/2} = \frac{ds}{dr},$$

$$\text{or} \quad \frac{dz}{dr} = \frac{\sqrt{2M}}{\sqrt{r - 2M}}, \qquad z = \sqrt{2M}\sqrt{r - 2M} + \text{const.},$$

so we have a paraboloid.

We see that the geometry can be weird, and that in particular the proper distance to the center (call it the radius) may increase faster than r = circumference $/2\pi$. In fact, if the spherical piece continued we sould find the surface of a spherical shell at radius s would approach a maximum and tehn decrease. The maximum is at $ds/dr = e^\Lambda = \infty$, or $2m = r$. Thus $dp/dr = \infty$, and in fact, not integrable, so this is impossible withing the confines of our static assumption.

0.17 Particle Orbits in Schwarzschild Metric

Consider a freefalling particle

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}.$$

We may choose our spherical coordinates such that, at one instant of time, $\theta = \pi/2$ and $d\theta/d\lambda = 0$.

$$\text{Then } \frac{d^2\theta}{d\lambda^2} = -2\Gamma^\theta{}_{\theta r}\underbrace{\frac{d\theta}{d\lambda}}_{0}\frac{dr}{d\lambda} - \underbrace{\Gamma^\theta{}_{\phi\phi}}_{-\frac12\sin 2\theta = 0}\left(\frac{d\phi}{d\lambda}\right)^2$$

so the orbit remains in the equatorial plane $\theta = \pi/2$, and $d\theta/d\lambda = 0$ always.

Let $p^\mu = \dfrac{dx^\mu}{d\lambda}$, which will be the momentum if $\lambda = \tau/m$. $p^\mu\partial_\mu$ is a vector. $p^\theta = 0$. Consider the claim that p_0 and p_ϕ are conserved:

$$\frac{d}{d\lambda}p_0 = \frac{d}{d\lambda}\left(g_{00}\,p^0\right) = \underbrace{\frac{g_{00}}{d\lambda}}_{g_{00,\mu}p^\mu} p^0 - g_{00}\underbrace{\Gamma^0{}_{\mu\nu}p^\mu p^\nu}_{2\Gamma^0{}_{0r}p^0p^r}$$

as the rest are zero.

But $g_{00}\Gamma^0{}_{0r} = \frac12 g_{00,r}$ so we get $g_{00,\mu}p^\mu p^0 - g_{00,r}p^0p^r = 0$, and p_0 is conserved.

$$\text{Similarly} \qquad \frac{dp_\phi}{d\lambda} = \underbrace{g_{\phi\phi,\mu}}_{0 \text{ unless } \mu = r \text{ or } \theta}\ \underbrace{p^\mu}_{0 \text{ if } \mu=\theta} p^\phi - \underbrace{\Gamma_{\phi\mu\nu}}_{\mu\nu=\phi r, r\phi} p^\mu p^\nu,$$

and in the second term $\Gamma_{\phi\phi r} = \frac12 g_{\phi\phi,r}$ so we get zero. Thus

$$\begin{aligned} p_0 &= -E = \text{constant} \\ p_\phi &= L = \text{constant} \end{aligned}$$

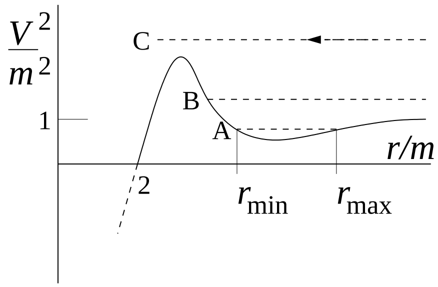
If the particle has mass, we choose $\lambda = \tau/m$. If not, choose λ so that p is the momentum, and set $m = 0$. Either way, $g^{\mu\nu}p_\mu p_\nu + m^2 = 0$. We need only consider the geodesic motion outside the star (as within an orbit is improbably, and not free falling!). So

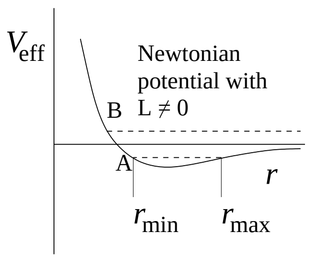
$$g^{00} = -\frac{1}{1 - \frac{2M}{r}}, \quad g^{rr} = 1 - \frac{2M}{r}, \quad g^{\phi\phi} = \frac{1}{r^2\sin^2\theta} = \frac{1}{r^2} \quad \text{on the orbit, and}$$

$$-\frac{E^2}{1-\frac{2M}{r}} + \left(\frac{dr}{d\lambda}\right)^2 \frac{1}{1-\frac{2M}{r}} + \frac{L^2}{r^2} + m^2 = 0,$$

or $\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V^2(r)$, where $V^2 := \left(1-\frac{2M}{r}\right)\left(m^2+\frac{L^2}{r^2}\right)$.

Case 1: $m \neq 0$
We now have a classical problem of a one dimensional motion in a potential. If E is at the level A, the particle moves in an orbit which comes in to $r_{\rm min}$, out to $r_{\rm max}$, and back again, repeatedly. In Newtonian physics this would be an elliptical orbit. If the energy is at B, we have an unbound orbit, with the particle coming in to $r_{\rm max}$ and then going out to infinity. This is a hyperbolic orbit in Newtonian physics. But a particle coming in with E at C crashes into $r=0$ even though $L \neq 0$, which Newton would never have.





The stable circular orbit occurs where

$$\frac{dV^2}{dr} = 0, \frac{d^2V^2}{dr^2} = 0. \qquad \text{Note} \quad \frac{1}{m^2}\frac{dV^2}{dr} = \frac{2M}{r^2} - \frac{2\tilde{L}^2}{r^3} + \frac{GM\tilde{L}^2}{r^4}$$

where $\tilde{L} = L/m$. Then $r_{\rm circle} = \dfrac{\tilde{L}^2 + \sqrt{\tilde{L}^4 - 12M^2\tilde{L}^2}}{2M}$.

If $\tilde{L}^2 < 12M^2$, there is no stable orbit. The particle will either escape to ∞ or crash into the center. The smallest stable circular orbit is at $r = 6M$.

The other root of the quadratic equation corresponds to an unstable orbit. For fixed M, the smallest r corresponds to $\tilde{L} \to \infty$, $r \to 3M$.

The orbit is governed by the equations

$$\begin{aligned} \frac{dt}{d\lambda} &= p^0 = g^{00}p_0 = \left(1-\frac{2M}{r}\right)^{-1} E \\ \frac{d\phi}{d\lambda} &= p^\phi = g^{\phi\phi}p_\phi = \frac{L}{r^2}, \qquad \theta = \frac{\pi}{2} \end{aligned}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V^2(r).$$

Example: suppose a particle is moving in a nearly circular stable orbit. Let $r_0 = \dfrac{\tilde{L}^2 + \sqrt{\tilde{L}^4 - 12M^2\tilde{L}^2}}{2M}$, and $r \approx r_0$. Thus $V^2(r) = V^2(r_0) + \frac{1}{2}(r - r_0)^2 \left.\frac{d^2V}{dr^2}\right|_{r_0}$. Let $k^2 = \frac{1}{2}\left.\frac{d^2V}{dr^2}\right|_{r_0} = m^2\left[-\frac{2M}{r_0^3} + \frac{3\tilde{L}^2}{r_0^4} - \frac{12M\tilde{L}^2}{r_0^5}\right]$. Now

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V^2(r) = \epsilon^2 - k^2(r-r_0)^2, \qquad \text{so} \quad r = r_0 + \frac{\epsilon}{k}\sin k\lambda.$$

Thus one trip out from the smallest radius (periastron or perihelion) to the next takes $\Delta\lambda = 2\pi/k$. During this time ϕ increases by

$$\begin{aligned} \Delta\phi &= \int_{-\pi/2k}^{3\pi/2k} \frac{d\phi}{d\lambda} d\lambda = \int \frac{L\, d\lambda}{(r_0 + \frac{\epsilon}{k}\sin k\lambda)^2} = \frac{2\pi L}{kr_0^2} + \mathcal{O}\left(\frac{\epsilon}{r_0 k}\right) \\ &= 2\pi\left[-\frac{2Mr_0}{\tilde{L}^2} + 3 - \frac{12M}{r_0}\right]^{-1/2}. \end{aligned}$$

But $\left(r_0 - \frac{\tilde{L}^2}{2M}\right)^2 = \frac{\tilde{L}^4}{4M^2} - 3\tilde{L}^2$, or $\tilde{L}^2 = \frac{Mr_0^2}{r_0 - 3M}$, and $\Delta\phi = 2\pi\left[1 - \frac{6M}{r_0}\right]^{-1/2}$.
For the Sun, $M_\odot = 1.99 \times 10^{30}$kg, $G = 6.673 \times 10^{-11}\text{m}^3/\text{kg s}^2 = 7.4248 \times 10^{-28}c^2\text{m/kg}$, so $M_\odot = GM_\odot = 1,477.5$ m.

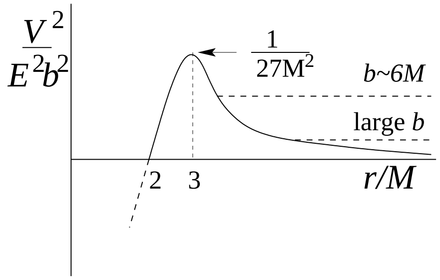
For Mercury, $r_0 = 5.79 \times 10^{10}$ m, so $\Delta\phi = 2\pi(1 + 3M_\odot/r_0) = 2\pi + 4.81 \times 10^{-7}$ rad. The perihelion shift ignores the 2π, so the shift is $\Delta\phi = 4.81 \times 10^{-7}$ rad/rev. Mercury revolves with a period of 7.60×10^6 s, so in a century it makes 415.2 revolutions, and

$$\Delta\phi = 1.997 \times 10^{-4}\,\text{rad/century} = 41.2\,\text{seconds of arc/century}.$$

Case 2: $m = 0$

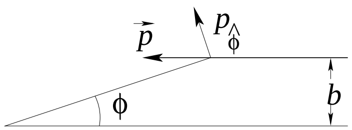
$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V^2(r),$$

$$V^2 = \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)$$



For a photon coming from infinity, with impact parameter b,

$$L = p_\phi = r p_{\hat\phi} = r|\vec p|\sin\phi = bE,$$



$$\text{so } \left(\frac{dr}{d\lambda}\right)^2 = E^2\left\{1\frac{b^2}{r^2}\left(1-\frac{2M}{r}\right)\right\}$$

$$\frac{d\phi}{d\lambda} = g^{\phi\phi}p_\phi = g^{\phi\phi}L = r^2 bE$$

$$\frac{dt}{d\lambda} = -g^{00}E = \frac{E}{1-2M/r}$$

Note under $\Lambda^* = E\lambda$, all equations lose dependence on E — the motion is independent of the energy of the massless particle.

Does a photon coming in get out again? The maximum of V^2/b^2E^2 is at $r = 3M$ with value $1/27M^2$, so for $E^2 > V^2_{\text{max}} = 27M^2b^2E^2$, or $b < 3\sqrt{3}M$, the photon will not turn around in r. That is, it will spiral in to $r = 0$, despite its having a nonzero L.

If $b > 3\sqrt{3}M$ there will be a turning point $r_{\rm T}$ for which $dr/d\lambda = 0$, at $r_{\rm T}^2 = b^2\left(1 - \frac{2M}{r_T}\right)$.

Consider a photon at a point r, with momentum $p_\mu = (-E, p_r, 0, p_\phi)$. Define the angle to the r axis by $\tan\delta = p_{\hat\phi}/p_{\hat r}$, where we use orthonormal coordinates

$$p_{\hat\phi} = \sqrt{g^{\phi\phi}}\,p_\phi = \frac{1}{r}p_\phi = \frac{bE}{r},$$

$$p_{\hat r} = \sqrt{g_{rr}}\,p^r = \left(1-\frac{2M}{r}\right)^{-1/2} E\left\{1 - \frac{b^2}{r^2}\left(1-\frac{2M}{r}\right)\right\}^{1/2},$$

so we see that

$$\tan\delta = \frac{b(r-2M)^{1/2}}{r^{3/2}}\left\{1 - \frac{b^2}{r^2}\left(1-\frac{2M}{r}\right)\right\}^{-1/2}.$$

A given angle determines b, and b at a given r determines whether the photon winds up at $r = 0$ or at $r = \infty$. Especially interesting is $r = 2M + \Delta r$, where Δr is small. Then escape requires $p_r > 0$ and $b > 3\sqrt{3}M$. The critical angle is therefore

$$\tan\delta = \frac{\sqrt{27}M(\Delta r)^{1/2}}{(2M)^{3/2}}\left(1 - \frac{27M^2}{8M^3}\Delta r\right)^{-1/2} \propto \sqrt{\Delta r},$$

so $\delta \propto \sqrt{\Delta r}$. A smaller and smaller cone of rays can get out as we approach $r = 2M$. We shall see later that within $r = 2M$, no light can ever escape.

A curious feature of this equation is that, for an incoming ray with b just slightly greater than $3\sqrt{3}\,M$, $dr/d\lambda$ becomes very small not only at the turnaround but for a long λ around it. If a roller coaster has just slightly too little energy to make the hill, it takes a long time to turn around. Here it is not time but λ which we have in our equation, but

$$\frac{dt}{d\lambda} = \left(1 - \frac{2M}{r}\right)E \sim \frac{1}{3}E \qquad \text{and} \qquad \frac{d\phi}{d\lambda} = \frac{L}{r^2} \sim \frac{bE}{9M^2}$$

are both perfectly nonzero, so ϕ and t both increase by arbitrarily large amounts as $b \to 3\sqrt{3}\,M$. The photon makes many orbits around the star before finally getting away!

0.18 An Isotropic Universe

The progress of physics was stymied for about 1700 years by the notion that the Earth was at rest in a preferred reference frame. When it was finally realized that this is not so, the reaction was very strong. With special relativity our prejudice that geocentrism is wrong and any frame is as good as another because dogma.

We therefore have a very strong feeling that one point in space is like any other (homogeneity), and one direction in space like another (isotropy). when it comes to cosmology, we must to some extent temper our statements about all Lorentz frames being like others, for there is clearly a frame in which the matter in a neighborhood of a few hundreds of millions of lightyears is at rest, the local rest frame.

We now know that the clusters of galazies are flying apart with a speed proportional to distance. Unless matter is being created (a violation of ${T^{\mu\nu}}_{;\nu} = 0$, hence "unthinkable"), this implies the universe was more dense in the past, so it is not true that one time is like any other. The best we can salvage seems to be this:

(a) On a sufficiently large scale, each point in space (each grand neighborhood really), in its local rest frame, looks the same in every spatial direction. (Isotropy)

(b) For to such pictures at the same time, the pictures are the same. That is, physics is independent of the particular spatial point. (Homogeneity)

What does "at the same time" mean? there exists a series of spacelike hypersurfaces which we may parameterize by t, such that for $t = c$, physics at $\vec{x}$ in the cosmological fluid's rest frame at $\vec{x}$ is the same regardless of choice of $\vec{x}$.

Consider the world lines of different elements of the fluid as we pass from one hypersurface t_1 to another t_2. The physics must be the same, so $\delta\tau$ is independent of $\vec{x}$. We might as well use proper time as t, so $g_{00} = -1$.

The velocity of the luid at $\vec{x}$ is $\mathbf{u} = \mathbf{d}t$. Let us establish three directions within the $t =$ constant hypersurface. Given a vector $\mathbf{v}$ in this hypersurface, $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot -\mathbf{v}$ by isotropy, so $g_{0j} = 0$.

Take a particulal "fiducial" hypersurface $t = t_0$. Erect a spatial coordinate system any way you like on this hypersurface, with metric

$$\mathbf{g}(\vec{x}) = \mathbf{d}t \otimes \mathbf{d}t + \gamma_{ij}(\vec{x})\mathbf{d}x^i \otimes \mathbf{d}x^j \qquad \text{for} \quad x^0 = t_0.$$

Let the coordinates of any event in spacetime by given by $x^0 = \tau$ measured along the fluid world line, and $x^j = x^j$ of the fluid world line when it passed through the fiducial hypersurface.

What is $\mathbf{g}$ at this point off the fiducial hypersurface? Consider the motions of all the neighboring peices of fluid. Their distance apart may hva changed, but it must have changed by the same amount in all directions, and angles must have stayed the same, by isotropy. Therefore $g_{ij}(t) = a^2(t)\gamma_{ij}$, and

$$(ds)^2 = -(dt)^2 + a^2(t)\gamma_{ij}\, dx^i \otimes dx^j.$$

Choice of the spatial coordinates:
Any point is like any other. Take a particular point and call it $\chi = 0$. Assign χ's to other points by the distance away from $\chi = 0$. Thus χ is a kind of radial coordinate.

Consider the two dimensional surface $\chi =$ constant. This, at least for small enough χ, must be a sphere. Choose one such sphere and erect spherical coordinates θ, ϕ on it. All directions on this sphere are isotropic, so $\gamma_{ij}dx^i dx^j = (d\chi)^2 + b^2(\chi)\left((d\theta)^2 + \sin^2\theta(d\phi)^2\right)$ by the same argument with which we decomposed $\mathbf{g}$ into $(dt)^2 + \gamma$. We extend θ and ϕ coordinates by geodesics. Ratios of distances must stay the same by isotropy. We choose b such that $b(0) = 0$, $b'(0) = 1$, so for sufficiently small circles the circumference $c = 2\pi\chi$.

We now calculate the curvature of the three dimensional space.

$$g_{ij} : g_{\chi\chi} = 1, \quad g_{\chi\theta} = g_{\chi\phi} = 0, \quad g_{\theta\theta} = b^2(\chi), \quad g_{\phi\phi} = b^2(\chi)\sin^2\theta$$

$$g_{ij,k} = 0 \text{ except } g_{\theta\theta,\chi} = 2bb', \quad g_{\phi\phi\chi} = 2bb'\sin^2\theta, \qquad g_{\phi\phi\theta} = b^2\sin(2\theta)$$

so

$${\Gamma^\chi}_{\theta\theta} = -bb', \quad {\Gamma^\chi}_{\phi\phi} = -bb'\sin^2\theta, \quad \text{other } {\Gamma^\chi}_{ij} = 0,$$

$${\Gamma^\theta}_{\theta\chi} = \frac{1}{2}g^{\theta\theta}g_{\theta\theta,\chi} = b'/b, \quad {\Gamma^\theta}_{\phi\phi} = \frac{1}{2}g^{\theta\theta}g_{\phi\phi,\theta} = -\frac{1}{2}\sin 2\theta,$$

$${\Gamma^\phi}_{\phi\chi} = b'/b, \qquad {\gamma^\phi}_{\phi\theta} = \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\theta} = \cot\theta.$$

$$D_\chi = \partial_\chi + \begin{pmatrix} 0 & 0 & 0 \\ 0 & b'/b & 0 \\ 0 & 0 & '/b \end{pmatrix}, \qquad D_\theta = \partial_\theta + \begin{pmatrix} 0 & -bb' & 0 \\ b'/b & 0 & 0 \\ 0 & 0 & \cot\theta \end{pmatrix}$$

$$D_\phi = \partial_\phi + \begin{pmatrix} 0 & 0 & -bb'\sin^2\theta \\ 0 & 0 & -\frac{1}{2}\sin 2\theta \\ b'/b & \cot\theta & 0 \end{pmatrix}.$$

$$\begin{aligned}
[D_\chi, D_\theta] &= \begin{pmatrix} 0 & -bb'' & 0 \\ b''/b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad R^\chi{}_{\theta\chi\theta} = -bb''; \quad R^\theta{}_{\chi\chi\theta} = +b''/b; \\
[D_\chi, D_\phi] &= \begin{pmatrix} 0 & 0 & -bb''\sin^2\theta \\ 0 & 0 & 0 \\ b''/b & 0 & 0 \end{pmatrix}; \qquad \begin{matrix} R^\chi{}_{\phi\chi\phi} = -bb''\sin^2\theta; \\ R^\phi{}_{\chi\chi\phi} = +b''/b; \end{matrix} \\
[D_\theta, D_\phi] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (1-b'^2)\sin^2\theta \\ 0 & b'^2-1 & 0 \end{pmatrix}; \qquad \begin{matrix} R^\theta{}_{\phi\theta\phi} = \sin^2\theta(1-b'^2; \\ R^\phi{}_{\theta\theta\phi} = b'^2-1 \end{matrix}.
\end{aligned}$$

$$\begin{array}{rclcccl}
 & & & (i=\chi) & (i=\theta) & (i=\phi) & \\
R &=& g^{jk}R^i{}_{jik} = & 0 & -b''/b & -b''/b & (j+\chi) \\
 & & & -b''/b & +0 & ?? & (j=\theta) \\
 & & & -b''/b & +(1-b'^2)/b^2 & +(1-b'^2)/b^2 & (j=\phi) \\
 &=& \multicolumn{5}{l}{-4\frac{b''}{b} + 2\frac{1-b'^2}{b^2}.}
\end{array}$$

Homogeneity requires this to be a constant, say $6K$.

Solutions:	(1) $b = K^{-1/2}\sin(\sqrt{K}\,\chi)$ for arbitrary K
$\sqrt{K} \to i\sqrt{K}$:	(2) $b = K^{-1/2}\sinh(\sqrt{K}\,\chi)$
$\lim_{K\to 0}$	(3) $b = \chi$

These are the only solutions to the homogeneous isotropic problem. They correspond to

1. A closed universe. the radii of the 2-spheres (surfaces of a ball) begin to shrink after $\chi > \pi/\sqrt{K}$, and shrink to zero at $\sqrt{k}\,\chi = \pi$. Thus χ only ranges up to $\pi/\sqrt{K}$, and the world has a finite volume

$$\begin{aligned}
V &= \int \sqrt{g}\,d\chi\,d\phi\,d\theta = \int b^2\sin\theta\,d\chi\,d\phi\,d\theta = 4\pi\int b^2 d\chi \\
&= 4\pi K^{-3/2}\int_0^{\pi/\sqrt{K}} \sin^2(\sqrt{K}\,\chi)\,\sqrt{K}\,d\chi = 2\pi^2 K^{-3/2}
\end{aligned}$$

   [This is easily pictured by surpressing one dimension as a surface of a sphere.]

2. An open universe. Here χ can grow without limit. The surfaces of spheres grow more rapidly than (radius)^2.

3. Flat space, with $\chi = r$ and the usual spherical coordinates.

In writting the 4-D metric, it is more convenient to scale χ, as there is a constant in front of γ_{ij} anyway. Let $\chi' = \sqrt{K}\,\chi$, so

$$\begin{aligned}
\gamma &= K^{-1}(d\chi')^2 + \sin^2\chi'\,d^2\Omega \qquad (K>0) \\
\gamma &= -K^{-1}(d\chi')^2 + \sinh^2\chi'\,d^2\Omega \qquad (K<0)
\end{aligned}$$

and

$$(ds)^2 - = -(dt)^2 + a^2(t)\left\{(d\chi)^2 + \underbrace{\begin{matrix}\sin^2\chi \\ \sinh^2\chi \\ \chi^2\end{matrix}}_{\text{choose world}}\left((d\theta)^2 + \sin^2\theta\,(d\phi)^2\right)\right\}$$

Einstein had a very strong prejudice in favor of a closed universe. Physicists, especially general relativists, have a very strong prejudice in favor of Einstein. So let us work out the case[11] for which the universe is closed, and use $\sin^2\chi$.

[This seems inappropriate to do:] We will accept the book's [MTW] calculation of G, as I'm sure you are not anxious for yet another curvature calculation. $G_{00} = 8\pi T_{00}$ gives

$$\frac{3}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{3}{a^2} = 8\pi\rho.$$

We need an equation for ρ, which can be a function of t. From $T^{0\nu}{}_{;\nu} = 0 = T^{0\nu}{}_{,\nu} + \Gamma^0{}_{\rho\nu}T^{\rho\nu} + \Gamma^\nu{}_{\rho\nu}T^{0\rho}$, $T^{00} = \rho$, $T^{0i} = 0$, $T^{ij} = p\,g^{ij}$, so we need $\Gamma^0{}_{ij}$ and $\Gamma^\nu{}_{0\nu}$.

$$\begin{aligned}
\Gamma^0{}_{\mu\nu} &= \frac{1}{2}g_{\mu\nu,0} = \frac{a'}{a}\left(g_{\mu\nu} + \delta^0_\mu\delta^0_\nu\right) \\
\Gamma^\nu{}_{0\nu} &= \frac{1}{2}g^{\nu\rho}g - \rho\nu, 0 = \frac{a'}{a}g^{ij}g_{ij} = 3\frac{a'}{a}
\end{aligned}$$

[11] But we now have new information that seems to favor a flat universe with a cosmological constant.

so

$$0 = \rho_{,t} + \frac{a'}{a}\left(g_{ij}T^{ij} + 3\rho\right) = \rho_{,t} + 3\frac{a'}{a}\,(p+\rho)\,.$$

Now we need to know p. Here we need an equation of state.
Case 1: Gas of matter with $m \gg T$.

Recall that for a gas of noninteracting particles $T^{\mu\nu} = \sum_n \int d\tau\, \delta^4(x^\mu - x^\mu_n) m u^\mu_n u^\nu_n$. In the rest frame of the fluid, therefore, $\rho = T^{00}\gamma^2 \approx m\gamma^2$ while $p = T_{ii} \approx mv^2\gamma^2$. If the particles are moving nonrelativistically with respect to the fluid "background" at their location, $\rho \gg p$, and we may ignore p in the conservation equation $\dfrac{1}{\rho}\dfrac{d\rho}{dt} = -\dfrac{3}{a}\dfrac{da}{dt}$, or $\rho \propto a^{-3}$.

It would be naïve to say "of course, the total energy of the universe is $\propto \rho a^3$ and is constant", because ρ is the matter energy density (without gravitational energy).

The situation is quite different if the particles are ultrarelativistic, *i.e.* with v very nearly equal to c with respect to the fluid rest frame. Photons and gravitons are always ultrarelativistic, and neutrinos almost always. Other particles are also relativistic if the temperature $T \gg 2m$. (We are choosing Boltzmann's constant $k_B = 1$. In conventional units we mean $k_B T \gg 2mc^2$.) There are no known particles which have small enough non-zero mass to satisfy this relation now, but near the beginning it was hotter. At $T \gg 10^10°$K, electrons were ultrarelativistic, and so on.

Suppose that the universe is dominated by radiation (or ultrarelativistic particles). Then

$$T = \sum \int E(1,\vec{v}) \times (1,\vec{v}) \quad \text{where} \quad v^2 = 1.$$

Averaging over directions, assuming isotropy of course, $\rho = 3p$, and

$$\frac{d\rho}{dt} = -\frac{3}{a}\frac{da}{dt} \times \frac{4}{3}, \qquad \text{or} \qquad \rho \propto a^{-4}.$$

Thus the energy density times volume is not conserved — rather it is inversely proportional to the size of the universe.

Consider what is happening. We have assumed the number of photons doesn't change, though not in a crucial way. If theere is relatively little matter around, a gas of photons will actually have the number of photons conserved because ther is not matter for them to interact with. Thus the energy per photon is falling with th expansion of the universe. That is, the wavelength

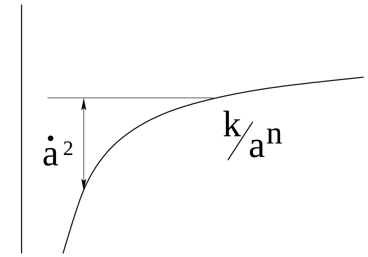
is proportional to the "radius" of the universe, just as if the photon were a standing wave in an adiabatically expanding container.

These two cases are extreme simplified cases, but they cover the recent history of the universe pretty well [NO — DARK ENERGY] Let us write $\rho \propto a^{-n}$, with $n = 3$ for a matter-dominated universe, $n = 4$ for a radiation dominated universe (and now $n = 0$ if dark energy dominates!).

Thus Einstein's equation tells us

$$\frac{3}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{3}{a^2} = \frac{k}{a^n}$$

or $\dot{a}^2 + 1 - \dfrac{k}{a^{n-2}} = 0$, (where we can replace the $+1$ with -1 for an open universe). As the universe expands, eventually k/a^n becomes 1 and $\dot{a} = 0$, thereafter it falls back in.



Thus the closed universe which led to our form of Einstein's equation is required to grow to a maximum size, and then collapse back down to zero.

If we had assumed the open universe with $\sinh^2\chi$ instead of $\sin^2\chi$, the sign of the $3/a^2$ term is reversed. When the universe is very young, a is very small, and the ± 1 is negligible. But as a gets large, the ± 1 term dominates. For the open universe $\dot{a}$ can never vanish, so the expansion continues forever.

At early times, we may write, approximating $\dot{a} = ka^{2-n/2}$, or $\dfrac{dt}{da} \propto a^{\frac{1}{2}n-1}$, $t \propto a^{n/2}$, or

$$a \propto t^{2/n}, \qquad \begin{cases} t^{2/3} & \text{matter} \\ t^{1/2} & \text{radiation} \end{cases}$$

This is the reason for the peculiar statement that, even in a closed universe which started at a point (of zero size), there are points of the universe we can not yet see. Consider a photon starting out as $\chi = 0$ at $t = 0$, travelling in the $\theta = 0, \phi = 0$ direcction. Then $\dfrac{d\chi}{dt} = \dfrac{1}{a(t)} \propto t^{-2/n}$, so $\chi \propto t^{1-2/n}$. Thus it has gone only a finite χ, (which is a form of spherical angle) since the beginning. If we are separated by more than the present $kt^{1-2/n}$ we can't see this event yet but will be able to later on.

By the way — you can't see the back of your head! If the photon left the back just after thebig bang, it will be approaching your eyeballs (along with everything else, but faster) just at the big crunch when the universe collapses to a point again.

Read Chapter 28 but skip box 28.1. You might also like to read Ch V of "The First Three Minutes".

Notice that Stephan-Boltzmann assures us that in the radiation dominated phase the energy density is $\rho = KT^4 \propto a^{-4}$, so $T \propto 1/a$. Therefore it was very hot at the beginning.

If we try to extrapolate the history of the universe back in time, say to time about 1 second, the temperature was so hot that all atoms were dissociated. In fact, even nuclei were dissociated. As time went on, $a \propto t^{2/n}$, $T \propto t^{-2/n}$, the temperature cooled to a point where free neutrons would decay, but also nuclei could become bound. This period, with $t \approx 2$ minutes, is when most of the deuterium and helium and lithium currently around was formed. After a brief period the nuclei were no longer sufficiently energetic to fuse.

the world was still too hot for atoms, and we had a plasma, in equilibrium with electromagnetic radiation. As the temperature dropped to three or four thousand degrees Kelvin, at about 10000 years, nuclei were able to bind electrons, and atoms formed and the plasma disappeared. Suddenly the universe became rather transparent to radiation, and so the black body radiation present at that time has continued, unscattered, to this day. However the wavelength has grown along with the universe about 1000 fold, so the temperature of this blackbody radiation has fallen to 2.7°K. The detection of this radiation by Penzias and Wilson led to the Nobel Prize in 1978, the year before I first prepared these notes.

The black body radiation is a great success of the Robertson-Walker metric. But it is also an embarrassment of sorts. Tests of the isotropy of the radiation have shown th temperature to vary by less than one part in a thousand in the local fluid rest frame. Thus the homogeneity of the fluid in the early universe was very good === all in equilibrium. But the universe we currently see consisted of many regions which had never had causal contact! How could this perfect homogeneity be established? This is called the horizon problem. A pssible solution comes from pushing even further back in time.

In the very early universe, the temperature was very hot and falling very quickly. At very high energy, the elementary physics is radically changed. For example, the objects which we consider particles are in fact composites of inseperable quarks. But at very high temperatures quarks are not expected to

be confined. This corresponds to a "false vacuum" state which has a different energy density from the true (temperature = zero) vacuum. At even higher temperatures, a grand unified symmetry may be restored, and the physics is best described as particles over a false vacuum of enormous energy density. As the universe cools, it would remain in this state for some time after the energy is supercooled. thus the major contribution to the energy density is just the extra energy density ρ_0, which this state contains, independent of a.

Then $\left(\frac{da}{dt}\right)^2 + 1 = \frac{8\pi}{3}\rho_0 a^2 + k' a^{-2}$, where ρ_0 is from the false energy of the vacuum and k' is from the excited quarks.

As a grows, the $\rho_0 a^2$ term dominates th 1 and the k'/a^2, and we have

$$\frac{da}{dt} = \sqrt{\frac{8\pi\rho_0}{3}}a, \qquad \text{or} \quad a(t) \propto e^{\sqrt{8\pi\rho_0/3}\,t},$$

growth at an exponential rate. But because the false vacuum is unstable, it will decay and turn its energy density into radiation, so eventually this exponential growth phase will stop, and lead into a more normal expansion of a radiation dominated universe.

Now if we extrapolate our present universe back in time to the beginning of the era of radiation domination of the current vacuum (at a time on the order of microseconds) the current universe consisted of causally disconnected patches. But if we extrapolate backwards the exponential growth, the size of this universe is much smaller, and could well have been causally connected and in equilibrium at very early times when SU(5) or some larger symmetry controlled the physics.

/subsectionThe Hubble Constant

Given two elements of fluid at separation $\Delta\chi$, the distance between them, $a(t)\Delta\chi$ is increasing at a rate $\dot{a}\Delta\chi$, so at early times

$$H = \frac{\text{velocity of separation}}{\text{separation}} = \frac{\dot{a}}{a} = \frac{2}{nt} \quad \text{is independent of}\,\chi.$$

It is not constant in t, and furthermore $H^{-1} = nt/2$ is an overestimate (by 50% if matter dominated) of the actual proper time. So if the Earth is ~ 5 billion years old (proper time),

$$H^{-1} \geq 7 \times 10^9 \text{ years}$$

or we have a contradiction. If some globular clusters in our galaxy are ten billion years old, as claimed by people who do stellar evolution studies, then $H \geq 15 \times 10^9$ years. Direct measurements of H yielded 18 billion in the mid '70's , but just as I was teaching this course for the first time, November '79, Walter Cronkeit announced it had fallen to 9-10 billion. The Rutgers Astrophysics group at the time said they didn't believe it.

There has been tremendous progress measuring the parameters of the universe since I first prepared these lectures. Much of this is based on precision measurements of the cosmic background radiation, the black-body radiation we mentioned earlier, but now its angular variation has been studied extensively. We also have much better measurements of distance versus redshifts to large distances. The current measurements point to about 13.5 billion years.

0.19 More on the Schwarzschild Geometry

Recall

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

What happens at $r = 2M$? The best way to find out is to go there. Might as well fall freely, radially,

$$\begin{aligned}
\frac{dt}{d\tau} &= \left(1 - \frac{2M}{r}\right)^{-1} \tilde{E}, \qquad \tilde{E} := E/m \\
\frac{dr}{d\tau} &= \pm\sqrt{\tilde{E}^2 - \tilde{V}^2} \qquad \tilde{V}^2 := V^2/m^2 = 1 - \frac{2M}{r} \quad \text{if } L = 0 \\
&= \pm\sqrt{\tilde{E}^2 - \frac{2M}{r} - 1}
\end{aligned}$$

We might as well fall in from infinity from rest, so E = energy at ∞, $= m$, and

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}, \quad \text{or} \quad \frac{r^{3/2}}{(2M)^{3/2}} = -\frac{\tau}{2M} + \text{const.}$$

We take infinitely long, of course, to fall in from infinity, but from any finite distance it takes only a finite proper time to pass through the mysterious $r = 2M$ and hit $r = 0$. Choose the constant to be zero (set clock).

What about t?

$$t = int \frac{1}{1 - \frac{2M}{r}} \frac{d\tau}{dr} dr,$$

but $\dfrac{d\tau}{dr} = -\sqrt{r2M} \to -1$ at $r = 2M$, so the integral diverges as $r \to 2M$, $\tau \to -2M$, $t \to 2M \int \dfrac{dr}{r - 2M} \sim -2M \ln\left(\dfrac{r}{2M} - 1\right) + \text{const.}$

What does this mean? Does it mean the falling physicist never gets to into teh Schwarzschild radius? Not at all, from his own point of view, he sails past without a glitch at $\tau = -2M$.

t is only a coordinate, not necessarily a good "time". But it is times in a sense. Outside the Schwarzschild radius, the paths of possible travel (forward light cone) are always with $\Delta t > 0$. So photons leaving the physicist and reaching an observer at ∞, or any other point outside $r = 2M$, do so after time t if they leave at coordinate t. Thus the observer at infinity, or even at some finite radius $> 2M$, never sees the free falling physicistenter the Schwarzshild radius.

The atoms in the physicists skin or spacesuit vibrate at their usual speed in proper time. But as we approach the Schwarzschild radius, this takes divergingly long in t. thus the outside observer sees the free falling physicist as redshifting away, barely visible just setting almost at the Schwarzschild radius.

I have spoken as if nothing unusual happens to the f.f.p. as he passes the S.R. But is not the metric singular there? The metric is a choice of coordinate system, not a physical quantity. What about the curvature? Components of the curvature tensor will be singular simply because our coordinate system is. To find out if something really terrible happens, we will attempt to set up a local inertial coordinate system. First start with the static orthonormal frame in which we calculated G. the curvature components in this frame are all $\propto 1/r^3$, with nothing funny at $r = 2M$. But this is a funny frame, for a particle sitting "at rest" at $r = 2M$ has $(ds)^2 = 0dt^2 + 0 = 0$, so it is lightlike (tracelling at the speed of light, in some sense). Let us instead choose a new $x^0 = \tau$ so $dx^0 = u$. the new orthonormal frame is related to the old (at $r > 2M$) by a Lorentz transformation. As r goes through $2M$, this transformation is still possible but becomes a Lorentz transformation faster than the speed of light. Thus the ffp inertial system and the Schwarzschild orthonormal fram are not both physical rest frames. Which is unphysical? The Schwarzschild, for at $r < 2M$, a particle "at rest" with $dr = 0$ has $(ds)^2 = -\left(1 - \frac{2M}{r}\right) dt^2 > 0$, spacelike, so is moving faster than light. There can be no such physical objects.

MTW assures us that in the local inertial frame

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -\frac{2M}{r^3} = -R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = 2R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = 2R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -2R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = -2R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}},$$

all others zero, so nothing singular happens at $r = 2M$

We are getting a strong impression that the singularity is not real, but a coordinate singularity. to get an idea about such things, consider a flat 3-space, which certainly has no physical singularities, in spherical coordinates.

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2, \qquad g^{\phi\phi} = \frac{1}{r^2\sin^2\theta},$$

singular both alond $r = 0$ and along $\theta = 0$.

If you are given a 3-D coordinate system, you expect $theta = 0$ is a plane. But it's not — $theta = 0$ is a line. What is $r = 2M$? in 4-D space we would expect it to be a hypersurface, with a three dimensional volume

$$\int \sqrt{g_{tt}g_{\theta\theta}g_{\phi\phi}}dtd\theta d\phi = 4\pi(2M)^2 \int \sqrt{g_{tt}}dt = 0$$

. So perhaps it is two dimensional. Surpressing θ, ϕ coordinates, which are not singular there except at $\theta = 0, \pi$, we have what we might expect to be a line (r =const, t arbitrary but finite) is really a point in 2-D spacetime.

There is a long motivation in the book, which will take you through Novikov coordinates, Eddington-Finkelstein coordinates, and then finally to the Kruskal-Szekeres coordinates,

$$\begin{aligned} u &= \sqrt{\frac{r}{2M} - 1}e^{r/4M}\cosh(t/4M) \\ u &= \sqrt{\frac{r}{2M} - 1}e^{r/4M}\sinh(t/4M) \end{aligned}$$

for $r > 2M$, which maps the region $r > 2M$ onto $u^2 - v^2 > 0$, $u > 0$. The "line" $r = 2M$, t finite is a mapped intot he point $(0, 0)$. Every particle falling into $r = 2M$ has $t = -2M\ln(r - 2M) + k$, so $\cosh(t/4M)$ and $\sinh(t/4M) \to \frac{1}{2}(\frac{r}{2M} - 1)^{-1/2}e^k$, so $u = v$ (a line) corresponds to the locus of points passed by different particles as $r = 2M$, more sensible than all particles falling in at $t = \infty$.

From $u^2 - v^2 = \left(\frac{r}{2M} - 1\right)e^{r/2M}$ we get $2udu - 2vdv = \frac{r}{(2m)^2}e^{r/2M}dr$. From $\frac{v}{u} = \tanh(t/4M)$ we get

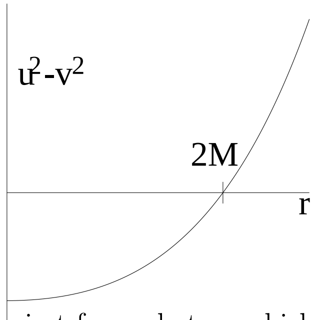
$$\frac{dv}{u} - \frac{v}{u^2}du = \frac{1}{4M}\frac{1}{\cosh^2(t/4M)}dt$$

or $udv - vdu = \frac{1}{4M}\left(\frac{r}{2M} - 1\right)e^{r/2M}dt$. Then

$$\begin{aligned} &-\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} \\ &= -(4M)^2\frac{2M}{r}\left(\frac{r}{2M} - 1\right)^{-1}(u\,dv - v\,du)^2 e^{-r/M} \\ &\quad +(4M)^2\frac{2M}{r}\left(\frac{r}{2M} - 1\right)^{-1}e^{-r/M}(u\,du - v\,dv)^2 \end{aligned}$$

$$= (4M)^2 \frac{2M}{r}\left(\frac{r}{2M}-1\right)^{-1} e^{-r/M}\left[(u^2-v^2)\,du^2-(v^2-u^2)dv^2\right]$$
$$= 32\frac{M^3}{r}\left(du^2-dv^2\right)e^{-r/2M}.$$

The singularity has disappeared.



Note r is now a function of u and v, given by

$$\left(\frac{r}{2M}-1\right)e^{r/2M}=u^2-v^2,$$

so r is a single valued function of u^2-v^2 until r gets to zero.

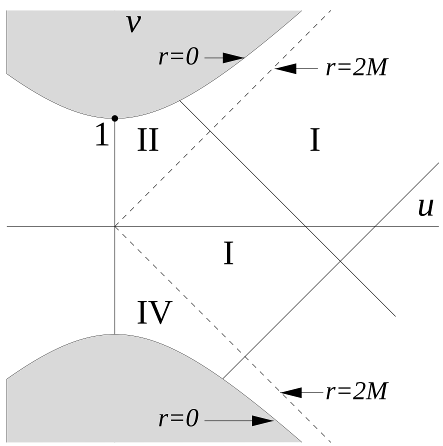
What happens to a particle falling in? It is easiest for a photon, which if radial ($d\theta = d\phi = 0$) satisfies $du^2 = dv^2$ just as in flat space. The solutions are

$$u=\pm v+k, \quad u^2-v^2=\pm 2vk+k^2=\left(\frac{r}{2M}-1\right)e^{r/2M}.$$

At large distances for finite k, $|t|$ must also get large, and we have

$$\pm 2k\sinh t/4M=\sqrt{\frac{r}{2M}-1}e^{r/4M}, \qquad \text{or} \quad \pm t\approx r \qquad (+\ln \text{ terms}).$$

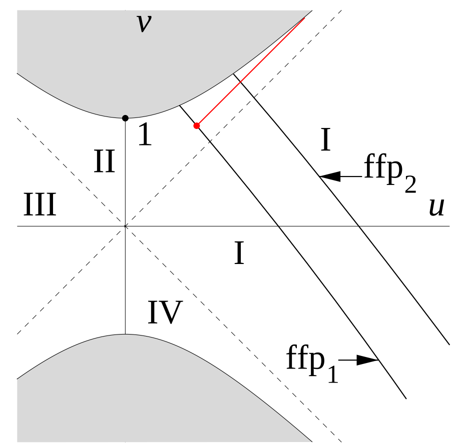
Thus the incoming photons are with a $-$ sign, outgoing with a $+$. The path of the photon is a straight line which eventually hits $v^2-u^2=1$ ($r=0$) where the curvature has a singularity. We cannot go past there. But nothing unusual happened at $r=2M$. The u, v coordinates continue right on in, although the connection with r and t no longer apply here. The analytic expressions give t an imaginary part, which, if ignored, gives the connection



$$u = \sqrt{1-\frac{r}{2M}}e^{r/4M}\sinh(t/4M)$$
$$v = \sqrt{1-\frac{r}{2M}}e^{r/4M}\cosh(t/4M).$$

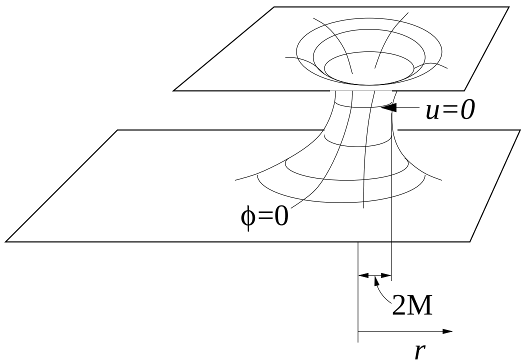
Consider an outgoing photon at $r>2M$. It has a line $u=v+k$. Perhaps it was created at an event with $r>2M$. But let us ask how far in it could have come from? As we push back further to $|u|=|v|$, $r=2M$, nothing singular occurs, and the photon might have come from even further in, right from $r=0$, $v^2-u^2=1$. The only difficulty is to see how anything could have got there to start it off, for no particle which was ever outside $r=2M$ can later be found in the region IV of the Kruskal-Szekeres diagram.

Consider our freely falling physicist (ffp) again. After passing $r=2M$, in a panic, he forgets his general relativity and tries to beam a message for help outward. the light ray, of course, does not escape, it eventually crashes into the hyperbola of the black hole. The photon is spied by another ffp, (#2), who chooses to spend his last microsecond speculating on how far back it could have come from. So he extrapolates it back, and finds that it might have come from outside the Schwarzschild radius. But



not from anyone he's ever talked to before, because the region III, $u<0, u^2>v^2$, although not separated from $u>0$ by a singularity, is not causally connected with any point in the region I. People in I and III can communicate only after both have irrevocably sacrificed themselves in the attempt.

Recall that we discussed embedding a star's geometry in a Euclidean 5 dimensional space. A snapshot at $t=0$ and at $\theta=\pi/2$ yields a tow dimensional surface in a 3-D Euclidean space, which was shaped inside the constant density model star like a section of a sphere, but outside like a parabola $z=\sqrt{8M}\sqrt{r-2M}$, from the geometry inside. Now our Schwarzschild geometry has the same form outside the star. Let us consider the spacelike hypersurface $v = 0$ which corresponds to $t = 0$. Then we are always outside the Schwarzschild radius, and we have:



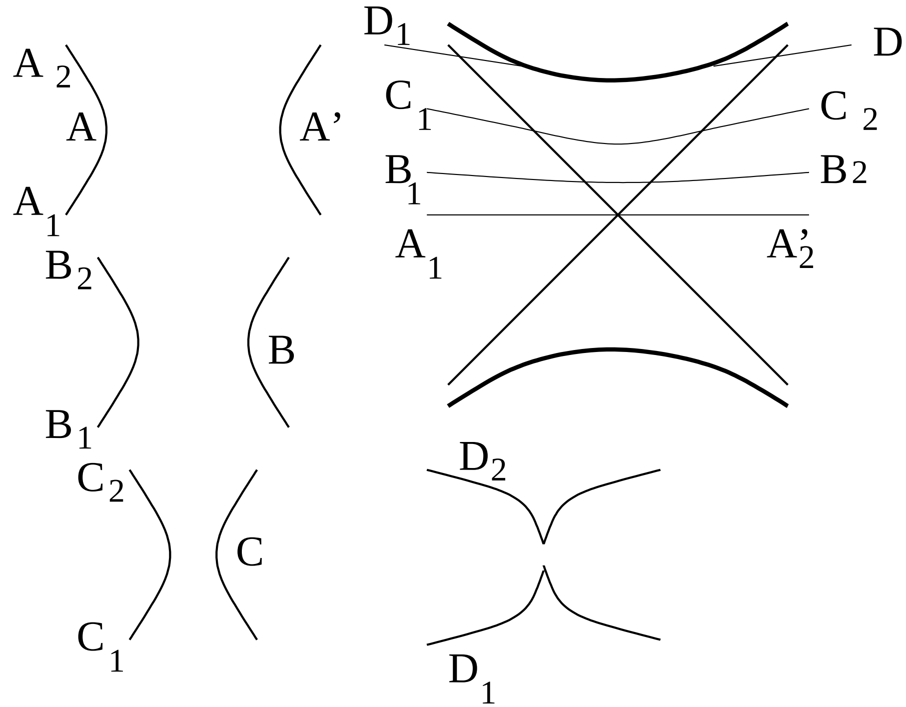
Away from the black hole there are two asymptotically flat spaces, $u > 0$ and $u < 0$. This is a spacelike hypersurface. It has no ends.

What happened to the piece of $t = 0$ $r < 2M$? It is not connected to $t = 0$, $r > 2M$. And it's not spacelike, as $g_{rr}dr^2 < 0$.

So it is $v = 0$ which is a complete snapshot of the universe(s), not $t = 0$. and it has two almost disconnected spaces.

Now we seem to have a paradox. The Schwarzschild geometry was static, so every slice in t generates a snapshot which looks like what we have here. Then why can't I simple walk along on of these $\phi =$ constant lines and get from one universe to another? The reason is that things aren't really static — the spacelike hypersurfaces $t =$ constant are, in the Kruskal system, lines through the origin. That is, the planes of different t do not permit the passage of time at $u = 0$, so successive smapshots will show an observer approaching but never reaching $u = 0$ ($r = 2M$).

We can, of course, propagate the hypersurface forwards in proper time everywhere. But we have now gone away from varying t at constant r. In particular, we have now included regions of $r < 2M$ in our spaceline hypersurface. Recall that the metric is not static in v, that is, $\mathbf{g}(u, v, \theta, \phi)$ **is** a function of v.



Successive hypersurfaces with a real lapse of proper time correspond to changing geometry, and bridge between the two universes are only connected for a short period.

Disclaimer: The Kruskal coordinate system and metric, with its two universes neatly joined at their singularity, is certainly a solution but probably not a physical one. Real black holes are formed by collapse, say of a star. Within the boundary layer of the star, the metric is no longer Schwarzschild or Kruskal, and the other universe doesn't exist.

But outside is still Schwarzschild. We derived S by asking for

(a) Solution to vacuum field equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ (for $r >$ surface)

(b) Spherical symmetry

(c) Static

In fact, Berkhoff's theorem states that (c) is unnecessary, that (a) and (b) are enough to give Schwarzschild in the vacuum region.

A proof is in the text (MTW) You'll have to read Box 23.3 to understand it (something we've skipped). I'm more interested in the motivation.

Consider an uncollapsed star, very slowly varying because of the light being given off, and non-rotating. Outside we have Schwarzschild. Now let it contract (maybe the core is used up, coools and collapses) to within the gravitational (Schwarzschild) radius. The geometry outside can only change if something tells it to, *i.e.* gravitational waves. And a spherically symmetric object and motion can't produce gravitational waves. (Easier problem to think about: collapsing spherically symmetric charge distribution) so the field outside, both gravitational and electromagnetic, doesn't change due to the collapse — it remains Schwarzschild.

That means that all the things we said about small particles falling into a static Schwarzschild singularity also hold for the gas on the surface of the star:

(a) No force (no $p(\rho)$ can stop the collapse once the surface is within $r = 2M$.

(b) Light emitted from the surface after that can't escape.

(c) An outside observer can never see the final collapse. Instead he just sees the surface fade away. In fact, the luminosity vanishes as $L \propto e^{-t/3\sqrt{3}M} = e^{-t/26\mu s}$ for a one-solar mass star. the light coming from the surface is similarly red-shifted $z = \frac{\Delta\lambda}{\lambda} = e^{t/4M}$ for radial photons,

though many of the photons arriving late are old photons coming from the surface as it passed $r = 3M$ and stored themselves in unstable orbits there. These have $z \approx 2$. Anyway, quibbling over redshifts in the last 100 μs of collapse seems silly right now.

0.20 Black Holes with Charge and Spin

In finding the Schqarzschild metric we insisted on

(a) Spherical symmetry, so $S = 0$

(b) no $T^{\mu\nu}$ outside, therefore no electric field, and $Q = 0$

Why are spin and charge important? Because you can detect the S and Q of a black hole from far away, by the dragging of inertial coordinates and by the electric flux. Baryon number, lepton number, *etc.*are not similarly measurable from afar.

The general solution with S and Q is

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right) - adt\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \tag{10}$$

where

$$a = S/M \quad \text{angular momentum / mass}$$

$$\begin{aligned}\rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - 2Mr + a^2 + Q^2\end{aligned}$$

This does not look very familiar. Note only that $\mathbf{g}$ is independent of ϕ, so it is axially symmetric. We must give $\mathbf{F}$ as well as $\mathbf{g}$.

$$\begin{aligned}\mathbf{F} = \frac{Q}{\rho^4}\Big\{&\left(r^2 - a^2\cos^2\theta\right)\mathbf{d}r \wedge \left(\mathbf{d}t - a\sin^2\theta\mathbf{d}\phi\right) \\ &+ 2ar\cos\theta\sin\theta\mathbf{d}\theta \wedge \left[\left(r^2 + a^2\right)\mathbf{d}\phi - a\mathbf{d}t\right]\Big\}\end{aligned} \tag{11}$$

We will not verify that this is a solution of the coupled Einstein and Maxwell equations, but accept it on faith. To understand the parameters, we note at large distances,

$$\begin{aligned}ds^2 = &-\left[1 - \frac{2M}{r} + \mathcal{O}(r^{-2})\right]dt^2 + 2a\left[1 - \frac{2M}{r} - 1\right]\sin^2\theta\, d\phi\, dt + \mathcal{O}(r^{-2}) \\ &+ \left[r^2\sin^2\theta + \mathcal{O}(1)\right]d\phi^2 + \left[r^2 + \mathcal{O}(1)\right]d\theta^2 + \left[1 + \mathcal{O}(r^{-1})\right]dr^2.\end{aligned}$$

Comparing to MTW 19.5 shows M is indeed the mass, and using $r^2\sin^2\theta d\phi = xdy - ydx$, we have $\sin^2\theta d\phi = \epsilon_{\ell j3}\frac{x^\ell dx^j}{r^2}$, with (?) $4aM = 4S$, so $a = S/M$.

The electric field is

$$\begin{aligned}\mathbf{F} &= \frac{\theta}{r^2}\left(\mathbf{d}r \wedge \mathbf{d}t + 2ar\cos\theta\sin\theta\mathbf{d}\theta\wedge\mathbf{d}\phi + \text{other components } \mathcal{O}(1)\right)\\ &= E_r\mathbf{d}r\wedge\mathbf{d}t + B_r r\mathbf{d}\theta\wedge r\mathbf{d}\phi + \text{other components },\end{aligned}$$

With $E_r = \dfrac{Q}{r^2}$, other components $\mathcal{O}(r^{-4})$, $B_r = \dfrac{2Qa}{r^3}\cos\theta$, so $p = Qa$ is the magnetic dipole moment.

The horizon:

Is it possible for a particle to maintain constant r? Its path must have $ds^2 < 0$, so this can only happen if $\Delta > 0$. This puts a horizon at $r^2 - 2Mr + a^2 + Q^2 = 0$, or

$$f = r_+ := M + \sqrt{M^2 - (a^2 + Q^2)}.$$

There is no horizon if $a^2 + Q^2 > M^2$.

We will later see that the area of the horizon is important. Lengths on a surface of fixed r, at fixed t, are $[dt = 0 = dr]$

$$ds^2 = \left[-\frac{\Delta a^2\sin^4\theta}{\rho^2} + \frac{(r^2+a^2)^2}{\rho^2}\sin^2\theta\right]d\phi^2 + \rho^2 d\theta??$$

so the area is $\int\left\{\rho^2\left[\dfrac{(r^2+a^2)^2}{\rho^2}\sin^2\theta - \dfrac{\Delta a^2}{\rho^2}\sin^4\theta\right]\right\}^{1/2} d\theta d\phi$. This simplifies on the horizon, where $\Delta = 0$, to

$$A = \int_0^\pi d\theta\int_)^{2\pi}(r_+^2 + a^2)\sin\theta = 4\pi(r_+^2 + a^2).$$

Once it becomes impossible for an observer to maintain his r, he is irrevokably doomed to fall into the hole, to $r = 0$. In some sense he is already there, for no photon he admits can get out either.

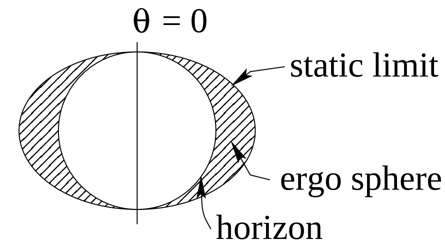
For a black hole with angular momentum, there is a dragging of the inertial frames. We saw this effect even in the weak field approximation, equation 19.5, but it becomes dramatic closer in, when it becomes impossible to "stay still". To stay put with respect to the background geometry $dr = d\theta = d\phi = 0$, so

$$ds^2 = \left(-\frac{\Delta}{\rho^2} + \frac{a\sin^2\theta}{\rho^2}\right)dt^2$$

had better be timelike (< 0). This "static limit" is at

$$\begin{aligned}\Delta &= a\sin^2\theta = (r-M)^2 + a^2 + Q^2 - M^2\\ \text{or}\qquad r = r_0(\theta) &= M + \sqrt{M^2 - Q^2 - a\cos^2\theta} \ge r_+.\end{aligned}$$

Within the ergosphere it is impossible not to rotate in the same direction as the hole. What happens to an infalling particle? Just as for the Schwarzschild case, the t coordinate goes to ∞ as the ffp reaches the horizon. But $d\phi/dt \to (r_+^2 + a^2)/a$, which is finite, so as $t \to \infty$ the orbit also makes infinitely many loops around during the plunge.



Yet all this happens in a finite proper time, with no singular tidal forces or torques on the ffp. One may make a change of variables to make the non-singular nature of the horizon apparent — the version in MTW is analogous to the infalling Eddington-Finkelstein coordinate system for the Scharzschild case, which we skimmed. Here, however, one needs to change the ϕ coordinate so we don't get all tangled up.

The Kerr-Newman black hole is a "static" solution, but we may ask what happens if an object falls in. There is, of course, conservation of momentum and energy (as understood in the asymptotically flat space), and also of angular momentum and charge. Actually an object won't put everything in, for in the process gravitational waves, and, if charged, Maxwell waves, are radiated, so not all the E, p, and L go in, but whatever doesn't get readiated away goes into changing the M, $\vec{v}$, $\vec{S}$ and Q of the black hole.

As

$$A = 4\pi(a^2 + r_+^2) = 4\pi\left(2M^2 - Q^2 + 2M\sqrt{M^2 - (a^2+Q^2)}\right),$$

a small change in M, Q and J $(= aM)$ gives

$$dM = \frac{\kappa}{8\pi}dA + \Omega dS + \Phi dQ.$$

(S is spin, not entropy) To evaluate κ, Ω and Φ, we write

$$\begin{aligned}
\frac{dA}{4\pi} &= 4MdM - 2QdQ + 2(r_+ - M)dM \\
&\qquad + \frac{2M}{r_+ - M}\left(MdM - QdQ - \frac{SdS}{M^2} + \frac{S^2}{M^3}dM\right) \\
&= \frac{2}{r_+ - M}\underbrace{\left[r_+^2 + a^2\right]}_{=A/4\pi} dM - \frac{2Qr_+}{r_+ - M}dQ - \frac{2a}{r_+ - M}dS,
\end{aligned}$$

$$\begin{aligned}
\text{so} \quad dM &= \frac{r_+ - M}{2A}dA + \frac{4\pi Q r_+}{A}dQ + 4\pi a A dS, \\
\text{so} \quad \kappa &= 4\pi\frac{r_+ - M}{A}, \qquad \Omega = \frac{4\pi a}{A}, \qquad \Phi = \frac{4\pi Q r_+}{A}.
\end{aligned}$$

This is to be thought of as a kind of first law of thermodynamics, with A playing the role of entropy and κ a temperature.

But this first law of holedynamics is not enough — things with black holes are not reversible. The second law is this: In any classical process the area of the horizon cannot decrease. We can increase or decrease the charge, the spin, and even the mass, but always subject to the constraint that the area of the horizon cannot decrease. One may even cosider colllisions of black holes, which may combine to form one whose area is at least as great as teh sum of the two initial ones. This is all supposed to remind you of entropy. That the black hole had entropy is not surprizing — after all, information about the infalling praticle is lost — only the energy, angular momentum and charge are remembered.

But if the black hole has entropy, in a cold empty space, shouldn't it radiate in a box with no entropy at all?

Classical mechanics does not permit this, and MTW only hints at an analogy. But since then it has been realized that quantum mechanics plays a vital role. Consider an $S = Q = 0$ hole, and let a pair of photons spontaneously emerge from the vacuum just outside $r = 2M$. Of course, there is not energy for two real photons, but if the one with negative energy falls into the hole within $\Delta t \sim \hbar/|E|$, who is to object? The remaining positive energy photon can then escape, with a net extraction of mass from the hole, and a shrinkage of the horizon.

The effect is that the black hole is a black body, and does radiate according to standard thermodynamics. It has a temperature

$$T = \frac{\kappa}{2\pi} = \frac{2(R_+ - M)}{A}.$$

Note for nonrotating uncharged holes $T \propto 1/M$. Thus $\frac{\partial M}{\partial T} < 0$, and black holes have a negative specific heat.

Large (starlike) black holes are so cold that their radiation is negligible. But it is possible that right after the big bang highly compressed regions of perhaps a billion tons of material formed little black holes with r_+ approximately 1 fermi (1 femtometer). Such objects could be found with a density of, say, 100 per cubic light-year, and would radiate 6,000 megawatts, a nice power source if you can find one.

Refs:
Hawking, Scientific American January 1977
Hawking, Phys Rev **D13** (76) 191.

0.21 Equivalence Principle, Fermions, and Fancy Formalism

We have discussed primarily how to treat the gravitational forces between objects. Other forces must be specified in other ways — general relativity does not tell you what these are, but it does specify how to incorporate them into curved space. A loose prescription is: convert all tensors in flat space to local tensors defined in terms of tangent vectors, and change all derivatives to covariant derivatives.

But how do you handle, for example, the Lagrangian for the electron,

$$\mathcal{L} = -\bar{\psi}\left(\gamma^\alpha \partial_\alpha - m\right)\psi \text{ ?}$$

Of course $\partial_\alpha \to D_\alpha$ but how does D act on ψ, which is not a tensor? And what about γ^α ? In special relativity γ^α are constant matrices, and ψ is neither a scalar nor a tensor, but a **spinor**. How does ψ transform under general coordinate chart change (coordinate transformation)? It cannot be defined!

Instead we must return to the equivalence principle. At a given event $\mathcal{P}$, set up a local inertial coordinate system ξ^α in a chart I not necessarily connected tot he x^μ which are the variables for chart M. Define the **vierbein** at that particular point to be

$$V^\alpha_{\ \mu} := \left.\frac{\partial \xi^\alpha}{\partial x^\mu}\right|_{\mathcal{P}}. \tag{12}$$

In the ξ coordinate system we have

$$\mathcal{L} = -\bar{\psi}(\xi)\left(\gamma^\alpha \frac{\partial}{\partial \xi^\alpha} - m\right)\psi = -\bar{\psi}(\xi)\left(\gamma^\alpha D_\alpha - m\right)\psi.$$

Changing to M, we find

$$\mathcal{L} = -\bar{\psi}(x)\left(\left(V^{-1}\right)^\mu_{\ \alpha}\gamma^\alpha D_\mu - m\right)\psi.$$

Notice that ψ is a spinor which transforms under changes of chart I. But chart I is restricted to orthonormal coordinates[12] at $\mathcal{P}$, so only Lorentz transformations are permitted. That's good, because I can define spinor representations

[12] More accurately, to a Minkowski space's coordinates.

of the Lorentz group, but I cannot define objects which transform like "spi-oors" under the group of arbitrary changes of chart at a point, general linear $4 \to 4$ real transformations.

The ψ's are scalar fields under changes of chart M. That is, $\psi'_a(x') = psi_a(x)$, as long as I hold I fixed. The γ's are still the numbers the field theory books tell me.

The vierbeins V have two indices but they live in different charts. Thus V^{-1} behaves like a *vector* in M in the index μ, and like a 1-form in I in the index α. Let's agree that M indices are $\mu, \nu, \rho, \ldots$ and I indices are $\alpha, \beta, \gamma, \ldots$. I indices are raised and lowered with the Minkowski metric $\eta_{\alpha\beta}$, M indices with $g_{\mu\nu}$, so

$$\begin{aligned} V_\alpha^{\ \mu} &= \eta_{\alpha\beta} V^\beta_{\ \nu} g^{\mu\nu} = \eta_{\alpha\beta} V^\beta_{\ \nu}\left(V^{-1}\right)^\nu_{\ \gamma}\left(V^{-1}\right)^\mu_{\ \delta}\eta^{\gamma\delta} \\ &= \eta_{\alpha\beta}\delta^\beta_\gamma \eta^{\gamma\delta}\left(V^{-1}\right)^\mu_{\ \delta} = \left(V^{-1}\right)^\mu_{\ \alpha}. \end{aligned}$$

ow we come to the crucial question: what does D_μ mean, acting on ψ. Recall that $D(object) = 0$ defines parallel transport of that object. If we parallel transport an I basis, we must be left with an orthonormal basis, so

$$(D_\mu \mathbf{e}_\alpha) = \omega_\mu^{\ \beta}{}_\alpha \mathbf{e}_\beta,$$

where the matrix $(\omega_\mu)^\beta_{\ \alpha}$ must be a generator of a Lorentz transformation, $(\omega_\mu)_{\alpha\beta} = (\omega_\mu)_{[\alpha,\beta]}$.

If I take a vector $\mathbf{u}$ and express it in the I chart,

$$(D_\mu \mathbf{u}) = D_\mu(u^\alpha \mathbf{e}_\alpha) = (\partial_\mu u^\alpha)\mathbf{e}_\alpha) + u^\alpha(\omega_\mu)^\beta$$

$$= u^\beta_{\ ;\mu}\mathbf{e}_\beta, \qquad \text{so} \quad u^\beta_{\ ;\mu} = u^\beta_{\ ,\mu} + \omega_\mu^{\ \beta}{}_\alpha u^\alpha.$$

Of course we may also express $\mathbf{u}$ and $D_\mu \mathbf{u}$ in the C chart $\mathbf{u} = u^\nu \mathbf{E}_\nu$, where the basis $\mathbf{E}_\nu$ in M is connected to $\mathbf{e}_\alpha$ in I by $\mathbf{E}_\nu V^\beta_{\ \rho}\mathbf{e}\beta$, so $u^\beta = V^\beta_{\ \nu} u^\nu$, $u^\nu = \left(V^{-1}\right)^\nu_{\ \beta} u^\beta = V_\beta^{\ /} nu u^\beta$,

$$\begin{aligned} D_\mu \mathbf{u} &= \left[u^\nu_{\ ,\mu} + \Gamma^\nu_{\ \rho\mu} u^\rho\right]\mathbf{E}_\nu \\ &= \left[\left(V_\beta^{\ \nu} u^\beta\right)_{,\mu} + \Gamma^\nu_{\ \rho\mu} V_\beta^{\ \rho} u^\beta\right] V^\gamma_{\ \nu}\mathbf{e}_\gamma \\ &= \left[u^\gamma_{\ ,\mu} + u^\beta \omega_\mu^{\ \gamma}{}_\beta\right]\mathbf{e}_\gamma \end{aligned}$$

so

$$\omega_\mu{}^\gamma{}_\beta = V_\beta{}^\nu{}_{,\mu} V^\gamma \nu + \Gamma^\nu{}_{\rho\mu} V_\beta{}^\rho V^\gamma_\nu.$$

What about ψ_a? We must have

$$\begin{aligned} (D_\mu \psi)_a &\equiv \psi_{a;\mu} = \psi_{a,\mu} + \Omega_{\mu a}{}^b \psi_b \\ \left(D_\mu \bar\psi\right)^a &\equiv \bar\psi^a{}_{;\mu} = \bar\psi^a{}_{,\mu} - \bar\psi^b \Omega_{\mu b}{}^a, \end{aligned}$$

so $D_\mu \bar\psi\psi = \partial_\mu(\bar\psi psi$ as we should have for a scalar.

Now $\bar\psi\gamma\psi$ is a vector, so

$$\begin{aligned} D_m u \bar\psi\gamma^\alpha\psi &= \partial_\mu(\bar\psi\gamma^\alpha\psi) + \omega_\mu{}^\alpha{}_\beta \bar\psi\gamma^\beta\psi \\ &= \bar\psi^a{}_{,\mu}\gamma^\alpha_{ab}\psi^b - \bar\psi^c \Omega_{\mu c}{}^a \gamma^\alpha_{ab}\psi^b + \bar\psi^c\gamma^\alpha_{cb}\left(\psi_{b,\mu} + \Omega_{\mu b}{}^a\psi_a\right), \end{aligned}$$

so $\omega_\mu{}^\alpha{}_\beta\bar\psi\gamma^\beta\psi = -\bar\psi\left[\Omega_m u, \gamma^\alpha\right]\psi$. The solution is $\Omega_{\mu a}{}^b = \frac{i}{4}\omega_\mu{}^\alpha{}_\beta \left(\sigma_\alpha{}^\beta\right)_a{}^b$, for

$$\begin{aligned} \left[\Omega_m u, \gamma^\alpha\right] &= \frac{i}{4}\omega_\mu{}^{\gamma\delta}\left[\sigma_{\gamma\delta}, \gamma^\alpha\right] \\ &= \frac{1}{2}\omega_\mu{}^{\gamma\delta}\left[\delta^\alpha_\gamma\gamma_\delta - \delta^\alpha_\delta\gamma_\gamma\right] \\ &= \frac{1}{2}\omega_\mu{}^{\alpha\delta}\gamma_\delta - \frac{1}{2}\omega_\mu{}^{\gamma\alpha}\gamma_\gamma = \omega_\mu{}^\alpha{}_\beta\gamma^\beta. \end{aligned}$$

Now we have three kinds of covariant derivatives:

$$D_\mu = \partial_\mu + A_\mu \qquad \begin{cases} A_\mu{}^\nu{}_\rho = \Gamma^\nu{}_{\rho\mu} \\ A_\mu{}^\alpha{}_\beta = \omega_\nu{}^\alpha{}_\beta \\ A_{\mu ab} = (\omega_\mu)_{ab} \end{cases}$$

And of course each generates an $R_{\mu\nu} = [D_\mu, D_\nu]$, with $(R_{\mu\nu})^\rho{}_\sigma \equiv R^\rho{}_{\sigma\mu\nu}$, $(R_{\mu\nu})^\alpha{}_\beta$ and $(R_{\mu\nu})^a{}_b$ all different matrices. But all of these matrices are representations of the Lorentz group, or rather, of generators of the Lorentz group. We may write the six generators as L_i and write

$$D_\mu = \partial_\mu + A_\mu{}^i L_i$$

where L_i is viewed as an abstract group generator. The $R_{\mu\nu}$'s are all essentially the same, for whatever the representation, the commutators of the L_i's are fixed by the structure of the group, $[L_i, L_j] = f_{ij}{}^k L_k$. Thus

$$\begin{aligned} \Gamma^\nu{}_{\rho\mu} &= A_\mu{}^i R_V(L_i)^\nu{}_\rho \\ \omega_\mu{}^\alpha{}_\beta &= A_\mu{}^i R_T(L_i)^\alpha{}_\beta \\ \Omega_\mu{}^a{}_b &= A_\mu{}^i R_S(L_i)^a{}_b \end{aligned}$$

but as they are all representations, $[R(L_i), R(L_j] = f_{ij}{}^k R(L-k)$.

We see that the covariant derivative is a description of how to do Lorentz transformationa while doing parallel transport. This is just as in a Yang-Mills theory, where the group is the Lorentz group. Thus it is a gauge theory.

One way of viewing a gauge theory, where fields depend not only on x^μ but also on some arbitrary choice of coordinates in the group space, is to treat the fields as functions of x and the choice of coordinates. We therefore have, say, $\psi(x, \Upsilon)$, where Υ represents a choice of coordinates. Let Υ_0 be a given one, then all the others are given by $g\Upsilon_0$, where g is the local symmetry group which relates all acceptible choices of coordinates. ψ, however, is not an arbitrary function of Υ — it transforms in a definite way — according to a specific representation $\psi(x, g\Upsilon_0) = R(g)\psi(x, \Upsilon_0)$.

The space of (x, g) on whih ψ depends is called a **fiber bundle**. The functions on this bundle which transform appropriately are, of course, much more limited than the general functions on this extended manifold. In fact, if for each x^μ there is some $g(x^\mu)$ for which one specifies the function $\psi(x^\mu, g(x^\mu))$, then ψ is determined everywhere. Such a cross section of the bundle $\sigma := (x^\mu, g(x^\mu))$ cannot be chosen in a special manner, so that same physics is described by another cross section $\sigma' = (x^\mu, g'(x^\mu))$ where $g'(x^\mu) = \Omega(x^\mu)g(x^\mu)$. The field ψ is reduced in one case to

$$\psi(x^\mu) := \psi(x^\mu, g(x^\mu))$$

while in the other to

$$\psi'(x^\mu) := \psi(x^\mu, g'(x^\mu)) = \mathcal{R}(\Omega(x^\mu))\psi(x^\mu).$$

The covariant derivative $D_\mu = \partial_\mu + A_\mu{}^i(x^\mu, g)L_i$ also has specified values $\mathcal{A}_\mu = A_\mu{}^i L_i$,

$$\begin{aligned} A_\mu{}^(x^\mu) \text{ on } \sigma \qquad & D_\mu\psi = \partial_\mu + \mathcal{R}(\mathcal{A}_\mu)\psi \\ A'_\mu{}^(x^\mu) \text{ on } \sigma' \qquad & D'_\mu\psi = \partial_\mu + \mathcal{R}(\mathcal{A}'_\mu)\psi \end{aligned}$$

$$\text{with} \quad D_\mu\psi|_{\sigma'} = \left(\partial_\mu + \mathcal{R}(\mathcal{A}'_\mu)\right)\mathcal{R}(\Omega)\psi = \mathcal{R}(\Omega)\left(\partial_\mu + \mathcal{R}(\mathcal{A}_\mu)\right)\psi$$

$$\text{or} \quad \mathcal{R}\left(\mathcal{A}'_\mu\right) = \mathcal{R}\left(\Omega\right))\,\mathcal{R}\left(\mathcal{A}_\mu\right)\mathcal{R}^{-1}\left(\Omega\right)$$

If the representation is faithful,

$$\mathcal{A}'_\mu = \Omega\mathcal{A}_\mu\Omega^{-1} - \left(\partial_\mu\Omega\right)\Omega^{-1}$$

specifies how gauge transformations proceed. It is, however, more transparent to write

$$D'_\mu = \Omega D_\mu \Omega^{-1}$$

which is equivalent as $\partial_m u(\Omega\Omega^{-1}) = 0$.

The usual formulation of gravity, in terms of $R^\mu{}_{\mu\rho\sigma}$, obscures this Lorentz group gauge invariance because

$$R^\mu{}_{\mu\rho\sigma} = \left(R^I_{\rho\sigma}\right)^\alpha{}_\beta V_\alpha{}^\mu V^\beta{}_\nu$$

is invariant under I Lorentz transformations.

Although the vierbein formalism permits us to cast gravity in a Yang-Mills type of formulation, the fact that the group is connected to space-time makes it different. In Yang-Mills, the Lagrangian density

$$\mathcal{L} \sim \sum_i F^i_{\mu\nu} F^{i\,\mu\nu},$$

which is not the action for gravity. This is because in addition to the connection $\omega_\mu{}^\alpha{}_\beta$ (which is analogous to A^i_μ) th vierbeins must enter into the action. It is

$$S = \int d^4x \left(-\det(V^\alpha_{\ \mu}\right) R_{\mu\nu\alpha\beta} V^{\alpha\mu} V^{\beta\nu}.$$

Back in the Dirac Equation, we can now write

$$\mathcal{L} = -\bar\psi \left(\gamma^\alpha D_\alpha - m\right)\psi$$

where $D_\alpha = V_\alpha{}^\mu D_\mu$. We may write

$$D_\alpha = V_\alpha{}^\mu \partial_\mu + \omega_\alpha, \qquad \omega_\alpha = V_\alpha{}^\mu \omega_\mu.$$

This anoholonomic covariant derivative has some interesting algebraic properties. Think of the fiber bundle as a 10 dimensional manifold, and the cross section as a four dimensional subsurface. A general change in σ generated by

$$\mathbf{\Omega} := \Omega^\mu \partial_\mu + \Omega^i L_i$$

can generate a new covariant derivative

$$D'_\alpha = \mathbf{\Omega} D_\alpha \mathbf{\Omega}^{-1}.$$

This corresponds to both a Lorentz gauge tranformation $\Omega^i L_i$ and a general coordinate transformation $x^\mu \to x^\mu + \Omega^\mu$.

The commutator of D's here is

$$[D_\alpha, D_\beta] = 2T_{\alpha\beta}{}^\gamma D_\gamma + R^i_{\alpha\beta} L_i.$$

In more general theories T is called the **torsion**. In general relativity $T = 0$, which imposes the condition which gives ω in terms of ∂'s and V's.

0.22 Quantized Field Theory

To write a quantized field theory, the basic ingredient normally is an action, the spacetime integral of a Lagrangian density. We treat the fields as operators.

Nearly all real results in field theory come from perturbation theory. The vacuum state is assumed to be nearly a simple state, and the actual fields are expanded about that simple state.

Whether this method is applicable to gravity is rather controversial, but we will outline it. we take as our "vacuum" flat space, and expand the metric (or, if we wish to include fermions, the vierbein) about its flat space value. We restore Newton's constant G for clarity

$$\mathcal{L} = -\frac{\sqrt{g}}{16\pi G} R + \mathcal{L}_{\rm matter}.$$

Define $\kappa = \sqrt{16\pi G}$, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. We use κh to give h the appropriate units for a boson field, with dimension $[h] = (L^{-1})$ in units with $\hbar = c = 1$, but $G \neq 1$.

$$\kappa = 1.15 \times 10^{-32}\,\mathrm{cm}$$

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2}\kappa \left(h_\rho{}^\mu{}_{,\nu} + h_\nu{}^\mu{}_{,\rho} - h_{\nu\rho,}{}^\mu \right) + \mathcal{O}(\kappa^2)$$

where indices on h are raised with η. Expanding the Lagrangian in powers of κ, we get
a kinetic energy term of $\mathcal{O}(\kappa^0)$ in h^2,
couplings of thre or more h's with powers of κ. The Lagrangian contains arbitrary orders in h due to the $\sqrt{g}$. Each such term corresponds to a possible vertex in a Feynman diagram, which itself represents a term in the expansion of $e^{iS/\hbar}$, evaluated between given initial and final states.

As you know, when you evaluate beyond the lowest order in perturbation theory, all field theories give divergent integrals, infinities in the evaluation of the amplitudes. It is possible to enumerate th possible operator forms which are multiplied by these divergent integrals just by dimensional analysis. For the action is dimensionless (or we couldn't exponentiate it). thus $\mathcal{L}$ has units of L^{-4}. Consider a theory such as ϕ^4, with

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

From the first term we see that, as $[\partial_\mu] = L^{-1}$, we have $[\phi] = L^{-1}$, and so $[m^2] = L^{-2}$ and $[\lambda] = L^0$.

In expanding the amplitude, we get extra terms of dimension 4. For example, divergent loop graphs may give

$$\left[\int \frac{d^4p}{p^2}\right] = L^{-2} \qquad \text{and} \qquad \left[\int \frac{d^4p}{p^4}\right] = L^0$$

which integrals are quadratically divergent and logarithmically divergent respectively. The only possible operators to multiply these by are
for quadratic divergence: ϕ^2
for logarithmic divergence: $\partial_\mu\phi\partial^\mu\phi$, $m^2\phi^2$ and ϕ^4.

All of the possible divergent forms are the same as objects already in the original Lagrangian, and can be taken care of by "renormalizing" the mass, coupling constant, and the strength of the kinetic energy term (the last called wave function renormalization).

For gravitation the situation is different. The coupling constant here is κ, which has dimension of length rather than being dimensionless. Therefore loop diagrams can generate terms of the form $\kappa^2 \left(\partial_\mu h \partial_\nu h \partial^\mu h \partial^\nu h\right)$ *etc.*, which are not contained in the original Lagrangian. Furthermore, as we go higher in orders of κ we generate infinitely many such terms, each in general diverging, adn so we might expect to have to specify an infinite nubmer of arbitrary renormalization constants to get rid of these infinities.

This very unpleasant circumstnce is known as **nonrenormalizablity**. Green's functions calculated to higher orders develop new types of infinities which must be subtracted out with arbitrary parameters to get finite Green's functions. This does happen in general relativity, and seems to make it a very unattractive theory.

It must be noted, however, that actual physical amplitudes are Green's functions evaluated on mass shell, that is, the operators ϕ which are left over after doing all Wick contractions are then describing particles obeying the classical equations of motion.

Now think of gravity at the one loop level, and let us write the operators which might diverge as tensors. Then to $\kappa^{-2}R$ is added a scalar of dimension L^{-4}, so to order κ^0 we may have

$$R^2, \quad R_{mu\nu}R^{mu\nu}, \quad \text{and} \quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}.$$

But the Gauss Bonnet theorem states

$$\int \sqrt{g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{mu\nu}R^{mu\nu} + R^2\right) = 0.$$

[This can be proven by writing $R_alpha\mu\nu\beta = [D_\mu, D_\nu]^\alpha{}_\beta$, using the Jacobi or Bianchii identities after integrating by parts.]

Thus any infinity multiplying $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ can be replaced by a combination of R^2 and $R_{mu\nu}R^{mu\nu}$. Now when we evaluate these operators for fields which satisfy the classical Einstein equations

$$R_{\mu\nu} = G_{\mu\nu} - \frac{1}{2}G_\mu{}^\mu g_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T_\mu{}^\mu\right), \qquad R = -T_\mu{}^\mu,$$

so if $T_{\mu\nu}$ vanishes, so to R and $R_{\mu\nu}$, and all the infinite terms in fact give zero contribution on shell.

Thus at the one loop level, at least, the quantum theory of gravity interacting only with itself gives finite physical scattering amplitudes. But if we have matter as well, this is not true — one loop amplitudes give infinities, even on shell in the presence of matter.

That is not much of a consolation. Our real world has eleeentary particles which are obviously not made up of gravitons, such as electrons, photons, and quarks, to name a few.

It has been explicitly checked that these possible infinities do in fact exist in theories of Einstein gravity coupled to

- scalars
- or photons
- or spin 1/2 fermions
- or Yang-Mills

The situation remained like this for many years, with most people believing the finiteness of pure gravity at one loop level to be of no consequence, because

- there is matter in the world
- what good is one loop finiteness if the higher loops are not finite. Are they? There are terms which might be generated in two loops (allowed by power counting and gauge invariance) which do not vanish on shell.

But then along came supergravity. Supergravity contains a symmetry group far bigger than the Lorentz group. In fact, it treats as essentially the

same fermions and bosons. Just as in SU(2) there are group generators L_i which rotate a π^+ into a π^0, in supersymmetry there are Q_i's which rotate, for example, quarks into gluons. Thus the Q changes fermions into bosons and vice versa. The anticommutators of the Q's give not only an ordinary (bose) symmetry operator L_i but also a piece proportional to the momentum generator. This means that the internal symmetry is tied up intimately with the space-time structure.

When one tries to make a gauge theory with supersymmetry (*i.e.* one tries to make supersymmetry a local symmetry, independently variable at every point in spacetime, just as for Yang-Mills or a photon's gauge invariance) one discovers that we are immediately forced into including a gauge field for the momentum operator. This is essentially the vierbein, and we are forced to include gravity as part of the gauge field. For greater definiteness I will first discuss a particular supersymmetry called O(4).

There are four Q_i's, each of which is a 4 component real Dirac spinor (a Majorana spinor). The theory contains a graviton with helicity 2. Applying each of the 4 Q_i's we lower the helicity and get four helicity 3/2 massless objects ψ_i. Applying another Q_i again, gives us 6 helicity 1 photon-like objects A_{ij}, antisymmetric in $i \leftrightarrow j$ because $\{Q_i, Q_j\} \propto P^\mu + L_{ij}$, neither of which has lower helicity. Continuing we get 4 helicity 1/2 objects and 1 helicity 0 object. The helicity -2 graviton generates a similar series, so we have a multiplet consisting of

- 1 graviton
- 4 Majorana spin 3/2 massless fields (gravitinos)
- 6 gauge vector particles
- 4 Majorana spin 1/2 particles
- 2 real scalars.

When we examine the forms of divergences in this theory, we find results exactly as in general relativity, as if the whole multiplet were a graviton. That is, the theory is formally nonrenormalizable but the one loop divergences vanish on mass shell, even in the presence of the helicity $\neq \pm 2$ objects in themultiplet. If we add a "matter" multiplet, with

- 1 helicity 1 particle

- 4 helicity 1/2 particles
- 6 helicity 0 particles
- 4 helicity -1/2 particles
- 1 helicity -1 particle

then the one-loop divergences no longer vanish on shell, just as in ordinary gravity.

Witheut the matter multiplet, one can even show that two-loop divergences cancel on-shell as well! So the situation is improved in two ways:

- Some of what we considered matter may in fact by part of the graviton multiplet
- Finiteness of the graviton multiplet is good through two loops, at least.

The O(4) theory without a matter multiplet has room for 6 vector mesons and 4 spinors, We need to include everything, so we need electrons, quarks, gluons, photons, $w^{\pm}$, *etc.*Remember the quarks come in 3 colors and 3 flavors, so 4 spinors is clearly not enough.

The largest supergravity theory of this conventional type is O(8), with 8 q_i's. The reason is that if there were nine, the state

$$Q_1 Q_2 \cdots Q_9 \,|\text{graviton with helicity } 2\rangle$$

would be a nonzero state with helicity $-5/2$, requiring a massless particle of spin at least 5/2. General relativity is the minimally complicated theory you can have for a spin 2 particle (except for one with no interactios) and spin 5/2 seems to be impossible.

What do we have in the O(8) theory?

Helicity	number	Physical Content	$SU(3)_C^Q$
2	1	graviton	
3/2	8	color triplet with charge $-1/3$	$3^{-1/3}$
		color antitriplet with charge 1/3	$\bar{3}^{1/3}$
		two chargeless singlets	1^0+1^0
1	28	gluons	8^0
		photon	1^0
		Z (weak neutral boson)	1^0
		lepto-quark gauge bosons	$3^{-1/3}+3^{-1/3}+\bar{3}^{1/3}+\bar{3}^{1/3}$
		diquark gauge bosons	$3^{2/3}+\bar{3}^{-2/3}$
1/2	56	u, d, s, c quarks	$2\times 3^{2/3}+2\times 3^{-1/3}$ + antiparticles
		electron + positron	$1^{+}1^{-}$
		neutrinos	1^0+1^0?
		other stuff	$8^0+8^0+6^{1/3}+\bar{6}^{-1/3}$

Is this enough? Nearly. We've got the gluons, photon, and neutral currents, but we have missed the $W^{\pm}$ which mediate ordinary weak interactions.

We have the quarks in 3 colors and 4 flavors, but no bottom or top. Worse yet, we have no *muon*. Close but no cigar!

Supergravitymay not be the grand unification Einstein hoped for, but not everyone has given up. For many years, 1930-1965 relativity was a dried up field. But now research has been reinvigorated by

- new technology making new tests possible
- new astronomy and cosmology, finding many conditions under which general relativity becomes important, even dominant.
- new ideas in combining quantum mechanics with general relativity — eventually there must be a reconciliation.

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