# Schwinger trick and Feynman Parameters



Here is the way Schwinger presented the method of combining propagators. An interesting anecdote of physics history is that Schwinger remained bitter that a virtually identical mathematical trick became commonly known as Feynman parameters. Why two brilliant physicists, each of whom had an appropriately won a Nobel prize, should fight over what is essentially a trivial mathematical trick, is an interesting question in the sociology of physicists. But we are not interested in that now.

Note that

$$\frac{1}{A} = \int_0^\infty d\nu \, e^{-A\nu},$$

at least when $\operatorname{Re} A > 0$, where the integral is well defined. Applying this to each of the terms in a product of inverses,

$$\prod_{i=1}^N \frac{1}{A_i} = \left( \prod_{i=1}^N \int_0^\infty d\nu_i \right) e^{-\sum\limits_{i=1}^N A_i \nu_i}.$$

Let $\nu = \sum \nu_i$ and $\alpha_i = \nu_i/\nu$. Then

$$\prod_{i=1}^N d\nu_i = \nu^{N-1} d\nu \prod_{i=1}^N d\alpha_i \, \delta\left(1 - \sum_{i=1}^N \alpha_i\right),$$

so

$$\prod_{i=1}^N \frac{1}{A_i} = \left( \prod_{i=1}^N \int_0^\infty d\alpha_i \right) \delta\left(1 - \sum_{i=1}^N \alpha_i\right) \int_0^\infty \nu^{N-1} d\nu e^{-\nu \sum \alpha_i A_i}.$$

But

$$\int_0^\infty t^{z-1} e^{-bt} dt = \frac{1}{b^z} \Gamma(z),$$

where

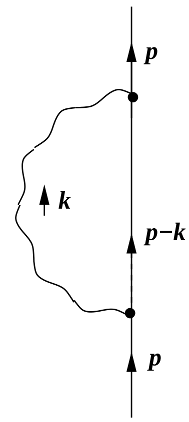
$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt = (z-1)!$$

is the Euler Gamma function. So all together

$$\prod_{i=1}^N \frac{1}{A_i} = \Gamma(N) \left( \prod_{i=1}^N \int_0^\infty d\alpha_i \right) \delta\left(1 - \sum_{i=1}^N \alpha_i\right) \left[\sum \alpha_i A_i\right]^{-N}. \qquad (1)$$

which is equation 6.41.

The simplest application would be something like




$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2 - m^2 + i\epsilon} \frac{1}{k^2 + i\epsilon}$$

which is something like what we would encounter if we cared about the second diagram of Fig. 6.1, which contains the (amputatable) correction shown. In terms of Feynman parameters, this is

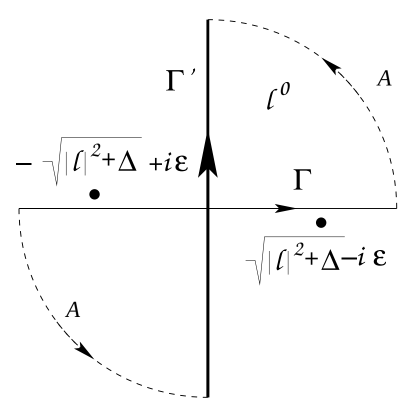
$$\begin{aligned}
\int \frac{d^4k}{(2\pi)^4} & \frac{1}{(p-k)^2 - m^2 + i\epsilon} \frac{1}{k^2 + i\epsilon} \\
&= \Gamma(2) \int_0^1 d\alpha \int_0^1 d\beta\, \delta(1-\alpha-\beta) \\
&\qquad \int \frac{d^4k}{(2\pi)^4} \frac{1}{(\alpha(p-k)^2 - \alpha m^2 + \beta k^2 + i(\alpha+\beta)\epsilon)^2} \\
&= \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - 2\alpha k\cdot p + i\epsilon)^2}
\end{aligned}$$

This can be simplified by noting that the denominator can be written as the square of $D$, where $D$ is defined in terms of a shifted four-momentum $\ell^\mu = k^\mu - \alpha p^\mu$ as

$$D = \ell^2 - \alpha^2 m^2 + i\epsilon = \ell^2 - \Delta + i\epsilon, \quad \text{with } \Delta = \alpha^2 m^2.$$

Thus our integral can be written as

$$\int_0^1 d\alpha \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta + i\epsilon)^n}$$




where the integrand is invariant under Lorentz transformations, and I have generalized 2 to an arbitrary power $n > \frac{1}{2}$. For a fixed $\vec{\ell}$, the $\ell^0$ integral is along the path $\Gamma$, but we can throw in the arcs $A$ at infinity, and then deform the integral to be along $\Gamma'$, as we are not passing any singularities. Writing $\ell^0 = iL^4$ and $\vec{\ell} = \vec{L}$, our integral is now

$$(-1)^n i \int_0^1 d\alpha \int \frac{d^4L}{(2\pi)^4} \frac{1}{(L^2+\Delta)^n},$$

where the integral is now over Euclidean space, with $L^2 = \sum_1^4 (L^i)^2$. In fact, let's generalize further to an arbitrary number of space-time dimensions $d$. As the integrand is rotationally invariant in $d$ dimensions, the angular integral $\int d\Omega_d$ is just the area $S_d$ of a unit $d-1$ sphere, where $S_d = 2, 2\pi, 4\pi, 2\pi^2$ for $d = 1, 2, 3, 4$ respectively. We will derive the general expression later. The full measure of integration is $dV = r^{d-1} dr d\Omega_d$ so we have

$$\int_0^1 d\alpha \int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell^2)^p}{(\ell^2 - \Delta(\alpha) + i\epsilon)^n} = \frac{(-1)^{n+p}\, i\, S_d}{(2\pi)^d} \int_0^1 d\alpha \int_0^\infty dL \frac{L^{d+2p-1}}{[L^2+\Delta(\alpha)]^n}.$$

This is treated in the book a bit later, with this equation as 7.80

In other notes[1] we show that

$$S_d = 2\,\frac{\pi^{d/2}}{\Gamma(d/2)} \qquad \text{and} \qquad \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}\, du = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

so

$$\begin{aligned} &\int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell^2)^p}{(\ell^2 - \Delta(\alpha) + i\epsilon)^n} \\ &\qquad = \frac{i\,(-1)^{n+p}\Gamma(\frac{1}{2}d+p)\Gamma(n-p-d/2)}{(4\pi)^{d/2}\Gamma(d/2)\Gamma(n)} \Big(\Delta(\alpha)\Big)^{\frac{1}{2}d+p-n}. \end{aligned} \tag{2}$$

[1] "$\Gamma(N/2)$ and the volume of $S^{D-1}$" and "The Beta function $B(x,y)$".