Lecture 23 Nov. 21, 2013

Photon Propagator, Renormalization of e

Read pages 219–222 (top).

In section 7.1, we find (7.31):

$$\delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dz \left[-z \ln \frac{z\Lambda^2}{(1-z)^2m^2 + z\mu^2} + 2(1-z)\frac{z(2-z)m^2}{(1-z)^2m^2+z\mu^2}\right].$$

In the last lecture, we found

$$\delta F_1(0) = \frac{\alpha}{2\pi}\int_0^1 dz(1-z)\left[\ln \frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2} + \frac{(1-4z+z^2)m^2}{(1-z)^2m^2+z\mu^2}\right],$$

so

$$\delta F_1(0)+\delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dz(1-2z)\ln\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2} + m^2\frac{(1-z)(1-z^2)}{(1-z)^2m^2+z\mu^2}.$$

In the first term integrate by parts, with $u = z(1-z)$, $v = \ln ...$, with $uv = 0$ at both endpoints, and

$$dv = \frac{1}{z} + \frac{2(1-z)m^2 - \mu^2}{(1-z)^2m^2+z\mu^2},$$

so

$$\begin{aligned} -\int u dv &= -\int_0^1 \left[(1-z) + z(1-z)\frac{2(1-z)m^2-\mu^2}{(1-z)^2m^2+z\mu^2}\right] \\ &= -\int_0^1 (1-z)\left[1 - 1 + m^2\frac{1-z^2}{(1-z)^2m^2+z\mu^2}\right], \end{aligned}$$

which cancels the second term, and

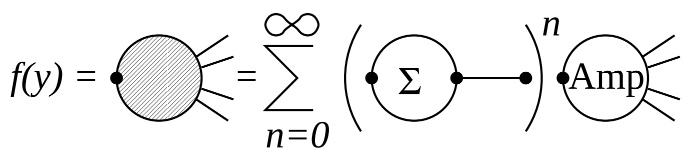
$$\delta F_1(0) + \delta Z_2 = 0.$$

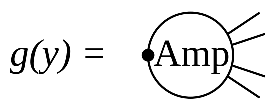
We are going to skip sections 7.2–7.4, but we need to make use of the main result of section 2, which is that the invariant amplitude $\mathcal{M}$ for any process is correctly given by the sum of amputated connected diagrams, but with a factor of $\sqrt{Z}$ for each external line.

A handwaving sketch of the derivation of this fact, given in section 2, is to ask how the fourier transform in x of a time ordered product involving $\phi(x)$ behaves near $p^2 = m^2$, where for simplicity I am taking a scalar field of physical mass m. On the one hand, we know that the time ordered product is given by the sum over *all* diagrams, so we have

$$\langle 0|\, T\phi(x)\ldots|0\rangle = \int dy D(x-y) f(y),$$

where





with $f(y)$ the sum of all diagrams (with the line to x removed) and $g(y)$ is the sum of diagrams with amputation on that leg.

$$\begin{aligned}
\langle 0|\, T\phi(x)\ldots|0\rangle &= \int dy D(x-y) f(y) \\
&= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_0^2 + i\epsilon} e^{-ipx} \tilde{f}(p) \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{p^2 - m_0^2 + i\epsilon} \sum_{n=0}^{\infty} \left(-i\Sigma(p^2) \frac{i}{p^2 - m_0^2 + i\epsilon} \right)^n \tilde{g}(p) \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{p^2 - m_0^2 - \Sigma(p^2) + i\epsilon} \tilde{g}(p)
\end{aligned}$$

The fourier transform will have a pole at $p^2 = m^2 = m_0^2 + \Sigma(p^2)$ and in the vicinity of that pole, we have

$$\begin{aligned}
\langle 0|\, T\phi(x)\ldots|0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{p^2 - m^2 - (p^2 - m^2)\dfrac{d\Sigma(p^2)}{dp^2} + i\epsilon} \tilde{g}(p) \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{iZ}{p^2 - m^2 + i\epsilon} \tilde{g}(p),
\end{aligned}$$

where

$$Z^{-1} = 1 - \left.\frac{d\Sigma(p^2)}{dp^2}\right|_{p^2=m^2}.$$

On the other hand, the time ordered product should be

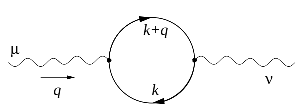
$$\langle 0|\,\phi(x)\,|p\rangle \frac{i}{p^2 - m^2 + i\epsilon}\mathcal{M},$$

and $\langle 0|\,\phi(0)\,|p\rangle = \sqrt{Z}$, so the invariant amplitude is given by $\sqrt{Z}\tilde{g}$, that is, the sum of all amputated diagrams with a factor of $\sqrt{Z}$ for each external leg.

Notice that now when we evalutate $F_1(0) = 1 + \delta F_1(0)$ we get

$$Z_2\Gamma^\mu(0) = Z_2F_1(0) = (1 + \delta Z_2 + \delta F_1(0))\,\gamma^\mu = \gamma^\mu.$$

Read pages 244–248 (top).



We turn to calculating Π, but in arbitrary space-time dimension d:

$$\begin{aligned} i\Pi_2^{\mu\nu}(q) &= i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2) \\ &= -4e^2\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \\ &\qquad \frac{2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}\left(m^2 + x(1-x)q^2\right)}{\left(\ell^2 + x(1-x)q^2 - m^2\right)^2} \end{aligned}$$

where we have shifted the integration variable and dropped terms linear in ℓ. As the book explains, this is not valid in four dimensions, because the integrals don't converge, but for small enough d this would be okay. We need to reexamine our treatment of $\ell^\mu\ell^\nu \sim \beta g^{\mu\nu}\ell^2$ for arbitrary d. As before, the integral vanishes for $\mu \neq \nu$ by antisymmetry in ℓ^μ, and is proportional to $g^{\mu\nu}$, but to get the proportionality constant β, note

$$g_{\mu\nu}\ell^\mu\ell^\nu = \ell^2 = \beta g_{\mu\nu}g^{\mu\nu}\ell^2 = \beta\, d\, \ell^2,$$

so $\beta = 1/d$. Had we had factors like $\gamma^\alpha\Gamma\gamma_\alpha$, we would have had to reevaluate those as well, replacing A.29 by A.55.

Thus, using Lecture 20 page 3, which says

$$\begin{aligned} I(d,p,n,\Delta) &:= \int \frac{d^d\ell}{(2\pi)^d}\frac{(\ell^2)^p}{(\ell^2-\Delta(\alpha)+i\epsilon)^n} \\ &= \frac{i\,(-1)^{n+p}\Gamma(\frac{1}{2}d+p)\Gamma(n-p-d/2)}{(4\pi)^{d/2}\Gamma(d/2)\Gamma(n)}\left(\Delta(\alpha)\right)^{\frac{1}{2}d+p-n} \end{aligned}$$

we can write

$$\begin{aligned} i\Pi_2^{\mu\nu}(q) &= -4e^2\int_0^1 dx\Big\{\left(\frac{2}{d}-1\right)g^{\mu\nu}I\left(d,1,2,\Delta(x)\right) \\ &\qquad +\Big[-2x(1-x)q^\mu q^\nu+g^{\mu\nu}\left(m^2+x(1-x)q^2\right)\Big]\,I\left(d,0,2,\Delta(x)\right)\Big\} \\ &= 4ie^2\int_0^1 dx\Big\{\left(\frac{2}{d}-1\right)g^{\mu\nu}\frac{\Gamma(\frac{1}{2}d+1)\Gamma(1-\frac{1}{2}d)\Delta(x)}{(4\pi)^{d/2}\Gamma(\frac{1}{2}d)\Gamma(2)} \\ &\qquad +\Big[2x(1-x)q^\mu q^\nu-g^{\mu\nu}\left(m^2+x(1-x)q^2\right)\Big]\frac{\Gamma(\frac{1}{2}d)\Gamma(2-\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(\frac{1}{2}d)\Gamma(2)}\Big\} \\ &\qquad (\Delta(x))^{\frac{1}{2}d-2} \end{aligned}$$

with $\Delta(x)=m^2-x(1-x)q^2$. Thus

$$\begin{aligned} i\Pi_2^{\mu\nu}(q) &= 4ie^2\int_0^1 dx\Big\{\left(\frac{2}{d}-1\right)\frac{d}{2}\left(m^2-x(1-x)q^2\right)g^{\mu\nu} \\ &\qquad +\Big[2x(1-x)q^\mu q^\nu-g^{\mu\nu}\left(m^2+x(1-x)q^2\right)\Big]\left(1-\frac{d}{2}\right)\Big\} \\ &\qquad \frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^2}\left(\frac{\Delta(x)}{4\pi}\right)^{\frac{1}{2}d-2} \\ &= \frac{i\alpha}{\pi}\Gamma\left(1-\frac{d}{2}\right)\left(1-\frac{d}{2}\right) \\ &\qquad \int_0^1 dx\left\{2x(1-x)(q^\mu q^\nu-q^2g^{\mu\nu}\right\}\left(\frac{\Delta(x)}{4\pi}\right)^{\frac{1}{2}d-2} \\ &= \frac{i\alpha}{\pi}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\left\{2x(1-x)(q^\mu q^\nu-q^2g^{\mu\nu}\right\}\left(\frac{\Delta(x)}{4\pi}\right)^{\frac{1}{2}d-2}. \end{aligned}$$

Notice that a number of "miracles" have occurred. First, while the integral converges for $d < 2$, the $\Gamma(1 - \frac{1}{2}d)$ blows up first at $d = 2$, when the ℓ^2 term first diverges. But the factor of $(2/d - 1)$ which multiples that, coming from $2\ell^\mu \ell^\nu - \ell^2 g^{\mu\nu}$, kills the divergence. Then the two terms combine in such a way that the separate $q^\mu q^\nu$ and $-q^2 g^{\mu\nu}$ terms develop the same coefficient, so that $\Pi^{\mu\nu}(q)$ has the correct prefactor, and we can write

$$\Pi(q^2) = -\frac{\alpha}{\pi}\Gamma\left(2 - \frac{d}{2}\right)\int_0^1 dx\, 2x(1-x)\left(\frac{\Delta(x)}{4\pi}\right)^{\frac{1}{2}d-2}.$$

Of course this expression still has a problem as $d \to 4$. Writing $d = 4 - \epsilon$, we have

$$\Pi(q^2) = -\frac{2\alpha}{\pi}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\, x(1-x)\left(1 - \frac{\epsilon}{2}\ln\frac{\Delta(x)}{4\pi}\right).$$

Using $\Gamma\left(\frac{\epsilon}{2}\right) \approx \frac{2}{\epsilon} - \gamma$, where $\gamma = 0.57721\ldots$ is the Euler (or Euler-Mascheroni) constant, and recalling that $\Delta(x) = m^2 - x(1-x)q^2$ we have

$$\begin{aligned}\Pi(q^2) &= -\frac{2\alpha}{\pi}\left\{\frac{2}{\epsilon} - \gamma - \ln\left(\frac{m^2}{4\pi}\right)\right\}\int_0^1 dx\, x(1-x) \\ &\quad + \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left(1 - x(1-x)\frac{q^2}{m^2}\right) \\ &= -\frac{\alpha}{3\pi}\left(\frac{2}{\epsilon} - \gamma - \ln\left(\frac{m^2}{4\pi}\right)\right) \\ &\quad + \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left(1 - x(1-x)\frac{q^2}{m^2}\right)\end{aligned}$$

Read pp 252–253, if we get that far.