Lecture 22 Nov. 18, 2013

Källen-Lehmann, Σ_2, Z and Z_2, $\delta F_1(0)$.

Last time we saw that the calculation of the first order (in α) correction to F_2 was untroubled by infrared or ultraviolet divergences, but the expression for the first order correction to F_1,

$$\delta F_1(q^2) = 2ie^2 \int \frac{d^4\ell}{(2\pi)^4} \int dx\, dy\, dz\, \delta(1-x-y-z) \frac{-\ell^2 + 2(1-x)(1-y)q^2 + 2(1-4z+z^2)m^2}{\left(\ell^2 - \Delta + i\epsilon\right)^3},$$

(with $\Delta = -xy\, q^2 + (1-z)^2 m^2$), diverges in the ultraviolet because of the term ℓ^2 in the numerator, and also in the infrared because Δ vanishes in the denominator at the $z \approx 1$ end of the integration interval. Last time we explained away the infrared divergence, and mentioned that $F_1(q^2) - F_1(0)$ doesn't have the ultraviolet divergence, but didn't really address why $F_1(0)$ is coming out wrong because of ultraviolet divergence.

Now we turn to understanding the ultraviolet divergence, and at the same time make more explicit the reason for amputating the feynman diagrams and the "one more modification" hinted at on p115. This is our introduction to the process of renormalization.

Read sections 7.1.

I have some notes expanding on the "Kinematics of p. 218" in the supplemental notes. This discusses 7.20 and the expression for k at the top of p. 219, and the discontinuity in p^2 of $\sigma_2(\not p)$

In section 7.1, we find (7.31):

$$\delta Z_2 = \frac{\alpha}{2\pi} \int_0^1 dz \left[-z \ln \frac{z\Lambda^2}{(1-z)^2 m^2 + z\mu^2} + 2(1-z) \frac{z(2-z)m^2}{(1-z)^2 m^2 + z\mu^2} \right].$$

In the last lecture, we found

$$\delta F_1(0) = \frac{\alpha}{2\pi} \int_0^1 dz (1-z) \left[\ln \frac{z\Lambda^2}{(1-z)^2 m^2 + z\mu^2} + \frac{(1-4z+z^2)m^2}{(1-z)^2 m^2 + z\mu^2} \right],$$

so

$$\delta F_1(0)+\delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dz(1-2z)\ln\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2} + m^2\frac{(1-z)(1-z^2)}{(1-z)^2m^2+z\mu^2}.$$

In the first term integrate by parts, with $u = z(1-z)$, $v = \ln ...$, with $uv = 0$ at both endpoints, and

$$dv = \frac{1}{z} + \frac{2(1-z)m^2-\mu^2}{(1-z)^2m^2+z\mu^2},$$

so

$$\begin{aligned} -\int udv &= -\int_0^1\left[(1-z)+z(1-z)\frac{2(1-z)m^2-\mu^2}{(1-z)^2m^2+z\mu^2}\right] \\ &= -\int_0^1(1-z)\left[1-1+m^2\frac{1-z^2}{(1-z)^2m^2+z\mu^2}\right], \end{aligned}$$

which cancels the second term, and

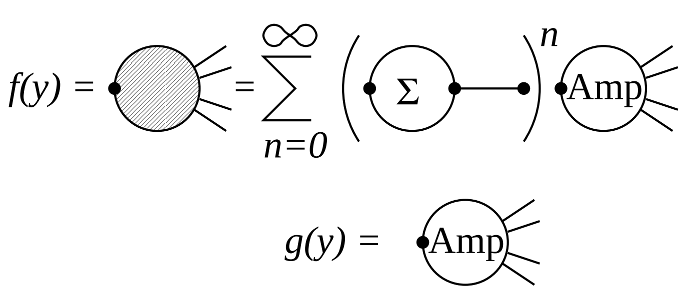
$$\delta F_1(0)+\delta Z_2 = 0.$$

We are going to skip sections 7.2–7.4, but we need to make use of the main result of section 2, which is that the invariant amplitude $\mathcal{M}$ for any process is correctly given by the sum of amputated connected diagrams, but with a factor of $\sqrt{Z}$ for each external line.

A handwaving sketch of the derivation of this fact, given in section 2, is to ask how the fourier transform in x of a time ordered product involving $\phi(x)$ behaves near $p^2 = m^2$, where for simplicity I am taking a scalar field of physical mass m. On the one hand, we know that the time ordered product is given by the sum over *all* diagrams, so we have

$$\langle 0|\,T\phi(x)\ldots|0\rangle = \int dy D(x-y)f(y),$$

where



with $f(y)$ the sum of all diagrams (with the line to x removed) and $g(y)$ is the sum of diagrams with amputation on that leg.

$$\begin{aligned}\langle 0|\, T\phi(x)\ldots|0\rangle &= \int dy D(x-y)f(y)\\ &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_0^2+i\epsilon}e^{-ipx}\tilde{f}(p)\\ &= \int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{i}{p^2-m_0^2+i\epsilon}\sum_{n=0}^{\infty}\left(-i\Sigma(p^2)\frac{i}{p^2-m_0^2+i\epsilon}\right)^n\tilde{g}(p)\\ &= \int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{i}{p^2-m_0^2-\Sigma(p^2)+i\epsilon}\tilde{g}(p)\end{aligned}$$

The fourier transform will have a pole at $p^2 = m^2 = m_0^2 + \Sigma(p^2)$ and in the vicinity of that pole, we have

$$\begin{aligned}\langle 0|\, T\phi(x)\ldots|0\rangle &= \int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{i}{p^2-m^2-(p^2-m^2)\dfrac{d\Sigma(p^2)}{dp^2}+i\epsilon}\tilde{g}(p)\\ &= \int \frac{d^4p}{(2\pi)^4}e^{-ipx}\frac{iZ}{p^2-m^2+i\epsilon}\tilde{g}(p),\end{aligned}$$

where

$$Z^{-1} = 1 - \left.\frac{d\Sigma(p^2)}{dp^2}\right|_{p^2=m^2}.$$

On the other hand, the time ordered product should be

$$\langle 0|\,\phi(x)\,|p\rangle\,\frac{i}{p^2-m^2+i\epsilon}\mathcal{M},$$

and $\langle 0|\,\phi(0)\,|p\rangle = \sqrt{Z}$, so the invariant amplitude is given by $\sqrt{Z}\tilde{g}$, that is, the sum of all amputated diagrams with a factor of $\sqrt{Z}$ for each external leg.

Notice that now when we evalutate $F_1(0) = 1 + \delta F_1(0)$ we get

$$Z_2\Gamma^\mu(0) = Z_2F_1(0) = 1 + \delta Z_2 + \delta F_1(0) = 1.$$