Lecture 15 Oct. 24, 2013

Fermion Potential, Yukawa and Photon Exchange

Last week we learned how to calculate cross sections in terms of the S-matrix or the invariant matrix element $\mathcal{M}$. Last time we learned, pretty much, how to evaluate $\mathcal{M}$ in terms of matrix element of the time ordered product of free fields (*i.e.* interaction representation fields) between states given naïvely by creation and annihilation operators for the incoming and outgoing particles.

Today we are going to address the first "real" applications.

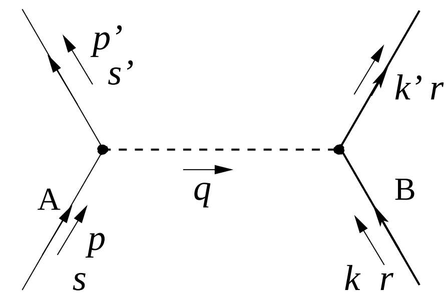
- First we will consider the low energy scattering of two different fermions each coupled to a single scalar field, as proposed by Yukawa to explain the nuclear potential[1].
- Then we will give Feynman rules for the electromagnetic field, that is, for photons. This will not be rigorously derived until next semester, but we can motivate a set of correct rules, which will get elaborated as we go on.
- Finally, we will couple the fermions with photons instead of Yukawa scalar particles, and see that this differs from the Yukawa case not only because the photon is massless, and therefore the range of the Coulomb potential is infinite, but also in being repulsive between two identical particles, and attractive only between particle - antiparticle pairs. We also see that gravity should, like the Yukawa potential, be always attractive.

Read Peskin and Schroeder pp. 118–126.

Let us consider the scattering of two distinguishable fermions

[1]This is often mentioned as a great success, with the known range of the nuclear potential predicting the existance of the pion. Of course we now know that the pion is not a scalar, but rather a pseudoscalar, with a coupling $ig\bar\psi\gamma^5\psi\phi$ instead of $g\bar\psi\psi\phi$. Then $\bar u(\vec p\,')\gamma^5 u(\vec p) \approx \chi^\dagger\vec\sigma\cdot(\vec p\,'-\vec p)\chi/(2m)$, which complicates the potential. It becomes spin dependent, but still, the factor $e^{-\mu r}/r$ emerges, modified by a quadratic expression in $(1/\mu r)$. These extra factors actually play an important role in the nuclear physics potential.

$$\langle 0|\, a_{B\,k'} a_{A\,p'} \bar\psi_B \psi_B \bar\psi_A \psi_A a^\dagger_{A\,p} a^\dagger_{B\,k}\, |0\rangle$$



$$i\mathcal{M} = i\frac{\bar u^{s'}(p',m_A)(-ig)u^s(p,m_A)\bar u^{r'}(k',m_B)(-ig)u^r(k,m_B)}{(p'-p)^2 - m_\phi^2 + i\epsilon}$$

Consider at low energy, $u^s(p,m_A) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar\sigma}\xi^s\end{pmatrix}$, $\gamma^0 = \begin{pmatrix}0 & 1\!\!1\\ 1\!\!1 & 0\end{pmatrix}$, $p^0 \sim m \gg |\vec p|$, so $\bar u^{s'}(p')u^s(p) = \xi^{s'\,\dagger}\left(\sqrt{p'\cdot\sigma}\sqrt{p\cdot\bar\sigma} + \sqrt{p'\cdot\bar\sigma}\sqrt{p\cdot\sigma}\right)\xi^s$, $\sqrt{p\cdot\sigma} = \sqrt{E}\sqrt{1-\vec v\cdot\vec\sigma} \to \sqrt{m}$, so $\bar u^{s'}(p')u^s(p) \sim 2m\xi^{s'\,\dagger}\xi^s = 2m\delta^{ss'}$.

$(p'-p)^2$ in the center of mass is $-(\vec p\,'-\vec p)^2$. $i\mathcal{M} = i\dfrac{(ig)^2}{-[(\vec p\,'-\vec p)^2+m^2]}$.

Do 4.123-4.127.

Now consider particle-antiparticle

$$\left\langle p'\bar k'\right|\bar\psi\psi\bar\psi\psi\left|p\bar k\right\rangle \sim \langle 0|\, b_{k'} a_{p'} \bar\psi\psi\bar\psi\psi a^\dagger_p b^\dagger_k\, |0\rangle$$

$$i\mathcal{M} = -ig^2\,\frac{-\bar u^{s'}(p')u^s(p)\bar v^r(k)v^{r'}(k')}{-(\vec p\,'-\vec p)^2 - m^2}$$

The minus sign in the numerator comes as follows: The two lowest shown contractions are both plus, which leaves $b\bar\psi\psi b^\dagger$, so the next higher contraction of $\bar\psi$ with $b^\dagger$ gives a minus, and then the last gives plus.

$$\bar v v = \xi^{r\,\dagger}\left(\sqrt{p\cdot\sigma}, -\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix}0 & 1\!\!1\\ 1\!\!1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{p'\cdot\sigma}\\ -\sqrt{p'\cdot\bar\sigma}\end{pmatrix}\xi^{r'} \to -2m\delta^{rr'}$$

Again $\mathcal{M} \to \dfrac{g^2}{\vec q^{\,2}+m^2}\delta^{rr'}\delta^{ss'}(2m)^2 \underset{\text{norm}}{\longrightarrow} \dfrac{g^2}{\vec q^{\,2}+m^2}\delta^{rr'}\delta^{ss'}$

So the Yukawa potential is attractive for particles and antiparticles in any combination, pp, $p\bar p$, or $\bar p\bar p$.

Note that actually the pi meson is actually a pseudoscalar. So the factors are changed

$$\bar u^{s'}(p')u^s(p) \quad\to\quad \bar u^{s'}(p')i\gamma^5 u^s(p) = \xi^{s'\,\dagger}\left(\sqrt{p'\cdot\sigma}, \sqrt{p'\cdot\bar\sigma}\right) i \underbrace{\gamma^0\gamma^5}_{\begin{pmatrix}0 & 1\!\!1\\ -1\!\!1 & 0\end{pmatrix}} \begin{pmatrix}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar\sigma}\end{pmatrix}\xi^s$$

$$= i\xi^{s'\dagger}\left[\sqrt{p'\cdot\sigma}\sqrt{p\cdot\bar\sigma} - \sqrt{p'\cdot\bar\sigma}\sqrt{p\cdot\sigma}\right]\xi^s,$$

Note to lowest order, $p \to (m,0,0,0)$ and this vanishes.

To next order, $\sqrt{p\cdot\sigma} = \sqrt{E}\sqrt{1-\vec v\cdot\vec\sigma} \sim \sqrt{E}\left(1-\frac{1}{2}\vec v\cdot\vec\sigma\right) \sim \sqrt{m}\left(1-\frac{1}{2}\vec v\cdot\vec\sigma\right)$ and $\sqrt{p\cdot\bar\sigma} \sim \sqrt{m}\left(1+\frac{1}{2}\vec v\cdot\vec\sigma\right)$ so

$$\begin{aligned}\bar u^{s'}(p')i\gamma^5 u^s(p) &\sim im\xi^{s'\dagger}\left[-\frac{1}{2}\vec v'\cdot\vec\sigma + \frac{1}{2}\vec v\cdot\vec\sigma - \left(-\frac{1}{2}\vec v'\cdot\vec\sigma + \frac{1}{2}\vec v\cdot\vec\sigma\right)\right]\xi^s \\ &= -im\xi^{s'\dagger}(\vec v' - \vec v)\cdot\sigma\xi^s \sim -im\xi^{s'\dagger}\vec q\cdot\sigma\xi^s.\end{aligned}$$

This complicates the form of the effective potential.

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After 4.133:

$$\begin{aligned}\bar u(p')\gamma^\mu u(p) &= \xi^{s'\dagger}\left(\sqrt{p'\cdot\sigma},\sqrt{p'\cdot\bar\sigma}\right)\begin{pmatrix}0 & 1\!\!1\\ 1\!\!1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar\sigma}\end{pmatrix}\xi^s \\ &= \xi^{s'\dagger}\left[\sqrt{p'\cdot\sigma}\bar\sigma^\mu\sqrt{p'\cdot\sigma} + \sqrt{p\cdot\bar\sigma}\sigma^\mu\sqrt{p\cdot\bar\sigma}\right]\xi^s \\ &\to \begin{cases} 2m\xi^{s'\dagger}\xi^s & \text{for } \mu = 0, \\ 0 & \text{for } \mu \neq 0\end{cases}\end{aligned}$$

so compared to Yukawa, we have -1 from g^{00}, repulsive, $\bar v\gamma^0 v \sim v^\dagger v = \xi^\dagger ... \xi \to +2m\delta^{rr'}$, so compared to Yukawa, we have a $-$ from $-g^{00}$ $-$ from $\bar v v$, so attractive.