Lecture 14 Oct. 21, 2013

Computing S-Matrix from Feynman Diagrams

From last time, we have for the general form of the cross section, Eq. 4.79,

$$\begin{aligned} d\sigma &= \frac{1}{2E_A 2E_B |\vec{v}_A - \vec{v}_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ &\qquad \times |\mathcal{M}(p_A, p_B \to \{p_f\})|^2 \, (2\pi)^4 \delta^4(p_A + p_B - \sum p_f), \end{aligned}$$

where

$$\begin{aligned} S((p_A, p_B \to \{p_f\}) &= {}_{\text{out}}\langle p_1 \ldots p_f | p_A p_B \rangle_{\text{in}} \\ &= \mathbb{1} + i(2\pi)^4 \delta^4(p_A + p_B - \sum p_f) \mathcal{M}(p_A, p_B \to \{p_f\}). \end{aligned}$$

For elastic scattering with all particles having the same mass, this reduces to

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{CM}}^2}.$$

Today we need to relate the invariant amplitude $\mathcal{M}$ to time ordered products of fields, along the lines we used in getting (the generalization of) 4.31:

$$\langle \Omega | \, T \left(\prod_j \phi(x_j) \right) | \Omega \rangle = \lim_{T \to \infty(1-i\epsilon)} \frac{\langle 0 | \, T \left(\prod_j \phi_I(x_j) \right) e^{-i \int_{-T}^{T} dt \, H_I(t) dt} | 0 \rangle}{\langle 0 | \, T e^{-i \int_{-T}^{T} dt \, H_I(t) dt} | 0 \rangle}$$

In terms of Feynman diagrams we saw that the ratio had the miraculous effect of cancelling the contribution of disconnected pieces, so that

$$\langle \Omega | \, T \left(\prod_j^n \phi(x_j) \right) | \Omega \rangle = \begin{array}{l} \text{sum of all (not necessarily fully) connected} \\ \text{diagrams with } n \text{ external points.} \end{array}$$

Read Peskin and Schroeder pp. 108–120.

When you get to fermions in section 4.7 in Peskin and Schroeder, I want to make some clarifications on spinors:

When we write down a sequence of fields, the order of the fields carries implied meanings. First, the fields are operators which act, starting on the right and then successively as we move left, on the states of the system. Secondly, the fields may be spinors, column or row vectors in the representation space, with implied contractions. For example, by $\bar\psi\psi$ we mean the single operator $\sum_a \bar\psi_a\psi_a$, but by $\psi\bar\psi$ we mean the four by four matrix operator $\psi_a\bar\psi_b$. This is not what is meant in 4.105, where we really have

$$T\left(\psi_a(x)\bar\psi_b(y)\right) \equiv \begin{cases} \psi_a(x)\bar\psi_b(y) & \text{for } x^0 > y^0 \\ -\bar\psi_b(y)\psi_a(x) & \text{for } x^0 < y^0 \end{cases} .$$

Note that S_F is a matrix:

$$S_{F\,ab}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not p + m)_{ab}}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)} = \langle 0|\, T\psi_a(x)\bar\psi_b(y)\,|0\rangle\,.$$

and the first displayed line on page 116 means

$$T\left(\psi_{1\,a}\psi_{2\,b}\psi_{3\,c}\psi_{4\,d}\right) = (-1)^3\psi_{3\,c}\psi_{1\,a}\psi_{4\,d}\psi_{2\,b} \qquad \text{if } x_3^0 > x_1^0 > x_4^0 > x_2^0.$$

If we put in all the spinor indices on all the ψ and $\bar\psi$ fields, as well as on any γ matrices Γ which might appear between them, the order in which we write down the Γ_{ab}'s and $S_{F\,cd}$'s are irrelevant, but if we arrange them in the right order, we can leave out the spinor indices as implied matrix multiplications. We find that when we draw the Feynman diagram for a given set of contractions, if we write the factors related to a single fermion line in order, starting from the head of the arrow which represents particle number flow, and working backwards to the start of that line, we get the factors in the correct order.