Lecture 11: Interacting Fields Oct 10, 2013

In the second problem of the first homework assignment, you explored the lagrangian density

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x), \quad \text{with} \quad F_{\mu\nu}(x) := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),$$

with the vector potential $A_\mu(x)$ as the four component dynamical field. You showed that the equations of motion then gave

$$\partial_\nu F^{\mu\nu} = \partial_\nu\partial^\mu A^\nu - \partial_\nu\partial^\nu A^\mu = 0,$$

which, with[1] $E^j = F^{j0}$ and $B^k = -\frac{1}{2}\epsilon_{ijk}F^{ij}$, gives Maxwell's equation in free space, without charges or current present. But in the project, problem 3.4d, as well as in lecture 8, PS 3.74, we learned that the Dirac field has a conserved current

$$J^\mu(x) = q\bar{\psi}(x)\gamma^\mu\psi(x)$$

so in homework 1, problem 3, we asked what happens to A_μ in the presence of a charge density J^0 and current density $\vec{J}$. Maxwell's equations for the fundamental fields E and B are, in rationalized MKS units, with $c = \epsilon_0 = \mu_0 = 1$, are

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= J^0, \\
\vec{\nabla}\cdot\vec{B} &= 0, \\
\vec{\nabla}\times\vec{E} + \dot{\vec{B}} &= 0, \\
\vec{\nabla}\times\vec{B} - \dot{\vec{E}} &= \vec{J}.
\end{aligned}$$

which corresponds to changing the equation of motion for $F^{\mu\nu}$:

$$\partial_\nu F^{\mu\nu} = J^\mu.$$

As you showed in the first homework, to get this equation of motion for A^μ, we need only add a term $-A_\mu(x)J^\mu(x)$ to the lagrangian:

$$\mathcal{L} = \mathcal{L}_0 - A_\mu(x)J^\mu(x) = -\frac{1}{2}\partial_\mu A_\nu(x)\partial^\mu A^\nu(x) + \frac{1}{2}\partial_\nu A_\mu(x)\partial^\mu A^\nu(x) - A_\mu(x)J^\mu(x).$$

[1] These signs are not generally agreed to. See "Notation Comparison", `convcomp.pdf`.

Then

$$\partial_\rho \frac{\delta \mathcal{L}}{\delta \partial_\rho A_\sigma} - \frac{\delta \mathcal{L}}{\delta A_\sigma} = -\partial_\rho \partial^\rho A^\sigma + \partial_\rho \partial^\sigma A^\rho + J^\sigma = -\partial_\rho F^{\sigma\rho} + J^\sigma = 0.$$

So we see that adding an interaction term

$$\mathcal{L}_I = -qA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$$

to the free Dirac and photon lagrangians will give us an interacting theory.

This interaction term is cubic in the fields, while up to now we have had only quadratic terms. Quite generally, terms in the lagrangian of higher order than quadratic in the fields give rise to nonlinear terms in the field equations, and nonlinear equations are hard to solve. Note that cubic and nonlinear here refer to the dependence on all the fields — it is not enough that the equations are linear in ψ for fixed A^μ, because the A^μ equations will depend on ψ. For each field, the free particles correspond to linear equations, and the nonlinear terms are responsible for interactions.

Read pages 77-87, though you should read 78 and the first half of 79 first.

Below are two discussions, first to clarify which fields are the arguments of H and H_0, and the second on the unitarity of $U(t,t')$ and 4.25.

Which fields, ϕ or ϕ_I, are arguments of H and H_0?

I had my troubles on pages 83-84, but I think I have resolved them. Here are the details of what was bothering me, and the resolution.

I had a problem with what H and H_0 represented. Are these are expressions in terms of $\phi(t,\vec{x})$ and $\pi(t,\vec{x})$, or in terms of $\phi_I(t,\vec{x})$ and $\pi_I(t,\vec{x})$? Here is how I resolved things.

4.12 is an expression in terms of the full fields, that is,

$$H(\phi,\pi,t) = H_{\text{Kl-G}}(\phi,\pi,t) + \int d^3x \frac{\lambda}{4!}\phi^4(t,\vec{x})$$

where

$$H_{\text{Kl-G}}(\phi,\pi,t) = \int d^3x \left(\frac{1}{2}\pi^2(t,\vec{x}) + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{x}))^2 + \frac{1}{2}m^2\phi^2(t,\vec{x})\right),$$

expressed in terms of the full fields. Note that $H_{\text{Kl-G}}$ depends on time through the fields, and is not independent of time, because it does not commute with the full Hamiltonian. On the other hand, the full hamiltonian H does commute with itself, so it is time independent, and the H's in $\exp \pm iH(t-t_0)$ in

$$\phi(t,\vec{x}) = e^{iH(t-t_0)}\phi(t_0,\vec{x})e^{-iH(t-t_0)}$$

can be evaluated at any time.

In defining the evolution of ϕ_I to be

$$\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)},$$

we take $H_0(t)$ to mean $H_{\text{Kl-G}}(\phi_I.\pi_I,t)$, not the Klein-Gordon hamiltonian in terms of the full field. Because ϕ_I evolves with H_0, $H_0(t)$ is time independent.

Then $U(t,t_0)$ is expressed in terms of both sets of fields,

$$U(t,t_0) = e^{iH_0(\phi_I,\pi_I,t_1)(t-t_0)}e^{-iH(\phi,\pi,t_2)(t-t_0)},$$

where t_1 and t_2 are arbitrary times, because each expression is time independent. The first line of 4.18 then becomes

$$\begin{aligned} i\frac{\partial}{\partial t}U(t,t_0) &= e^{iH_0(\phi_I,\pi_I,t_1)(t-t_0)} \\ &\qquad [H(\phi,\pi,t_2) - H_0(\phi_I,\pi_I,t_1)] \\ &\qquad e^{-iH(\phi,\pi,t_2)(t-t_0)}. \end{aligned}$$

What had me bothered is that the term in brackets does not look like H_{int}, because the H and H_0 are evaluated with different fields, while in 4.12 they are both evaluated with the full field. But they are independent of t_1 and t_2, so I can choose both to be t_0, in which case $\phi(t_0,\vec{x}) = \phi_I(t_0,\vec{x})$ and similarly for π, so the bracket is, in fact, $H_{\text{int}}(\phi_I,\pi_I,t_0)$. From the last line of 4.18 (and 4.19) we also see that H_I is to be interpreted in terms of ϕ_I, because it evolves with H_0.

Unitarity of $U(t,t')$ and Eq. 4.25

How to show 4.25 directly was not obvious to me, but if we first show $U(t,t')$ is unitary, and 4.26, then it follows easily.

As hermitian conjugation reverses the order of operators,

$$U^\dagger(t,t') = T^{-1}\left\{\exp\left[i\int_{t'}^{t} dt'' H_I(t'')\right]\right\},$$

where I have used[2] T^{-1} as the anti-time-ordering operator. Therefore the derivative with respect to t brings down a factor on the right:

$$\frac{\partial}{\partial t}U^\dagger(t,t') = iU^\dagger(t,t')H_I(t),$$

so $\dfrac{\partial}{\partial t}U^\dagger(t,t')U(t,t') = U^\dagger(t,t')\left(iH_I(t) - iH_I(t)\right)U(t,t') = 0$, and as $U(t',t') = 1$, we have $U^\dagger(t,t')U(t,t') = 1$ for all t, and $U(t,t')$ is unitary.

The first of equations 4.26 is obvious in terms of the time ordering expression, as all the times $\in [t_1,t_2]$ are later than those in $(t_2,t_3]$. The second equation is then the first, after multiplying by $U(t_2,t_3)$ on both sides. Defining $U(t_2,t_3) = U^{-1}(t_3,t_2)$ for $t_3 > t_2$) removes the $t_1 \geq t_2 \geq t_3$ restriction on 4.26.

Finally,

$$\begin{aligned} U(t,t') &= U(t,t_0)U(t_0,t') = U(t,t_0)U^\dagger(t',t_0) \\ &= e^{iH_0(t-t_0)}e^{-iH(t-t_0)}e^{iH(t'-t_0)}e^{-iH_0(t'-t_0)} \\ &= e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)}, \end{aligned}$$

which is 4.25.

[2]This is *ad hoc*: don't assume anyone will understand T^{-1} outside this discussion.