Lecture 9 Oct. 3, 2013

Quantization of the Dirac Field

Last time we found the Lagrangian density for a Dirac field ψ, which is a four component complex field,

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi.$$

Thus the canonical momentum conjugate to ψ is

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger,$$

so the $p\dot{q}$ term in $H = \sum p\dot{q} - L$ just cancels off the time piece, and the Hamiltonian density is

$$\begin{aligned} H &= \int d^3x\, \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi \\ &= \int d^3x\, \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta)\psi, \end{aligned}$$

in terms of the $\vec{\alpha} := \gamma^0\vec{\gamma}$, $\beta = \gamma^0$ used in one-particle quantum mechanics, where the operator hamiltonian is

$$h_D = -i\vec{\alpha}\cdot\vec{\nabla} + m\beta.$$

We will skip pages 52-57, which explains what goes wrong if we try to treat the field ψ as an almost commuting c-number field, in the way we did for the Klein-Gordon field ϕ. I am not going to review those reasons, but just present that instead, we need to treat ψ as an almost *anti-commmuting* object. That is, in the classical approximation we have for two components of the fields at different points

$$\psi_a(\vec{x})\psi_b(\vec{y}) = -\psi_b(\vec{y})\psi_a(\vec{x}),$$

and similarly for ψ with $\psi^\dagger$ and $\psi^\dagger$ with itself. That is to say, classically the **anti-commutator**,

$$\{A, B\} := AB + BA$$

of two Dirac fields vanishes. Quantum mechanically these are modified, just as for ordinary (bose) degrees of freedom momentum and position fail to

commute. Thus the anti-commutation relations (at equal times) are modified to

$$\begin{aligned} \{\psi_a(\vec{x},t), \psi_b(\vec{y},t)\} &= 0 \\ \{\psi_a(\vec{x},t), \psi_b^\dagger(\vec{y},t)\} &= \delta_{ab}\delta^3(\vec{x}-\vec{y}) \\ \{\psi_a^\dagger(\vec{x},t), \psi_b^\dagger(\vec{y},t)\} &= 0 \end{aligned}$$

A complete set of classical solutions of the Dirac equation are given by

$$u^s(\vec{p})\, e^{-ip\cdot x} \quad \text{and} \quad v^s(\vec{p})\, e^{ip\cdot x} \quad \text{for all} \quad \vec{p}, s = 1,2, \quad p^0 > 0,$$

with $u^s(\vec{p})$ and $v^s(\vec{p})$ given by 3.59 and 3.62, and satisfying 3.66 and 3.67. Thus we can expand the quantum field ψ in terms of these, with coefficients $a^s_{\vec{p}}$ and $b^{s\,\dagger}_{\vec{p}}$, which will become operators upon quantization. Thus

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s \left(a^s_{\vec{p}} u^s(\vec{p})e^{-ip\cdot x} + b^{s\,\dagger}_{\vec{p}}\, v^s(\vec{p})e^{ip\cdot x}\right) \tag{1}$$

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s \left(b^s_{\vec{p}} v^{s\,\dagger}(\vec{p})e^{-ip\cdot x} + a^{s\,\dagger}_{\vec{p}} u^{s\,\dagger}(\vec{p})e^{ip\cdot x}\right) \tag{2}$$

(with $p^0 = +\sqrt{\vec{p}^{\,2} + m^2}$ understood)

We may see that the anticommutation relations for the fields (3.102) will be obtained if the creation and annihilation operators satisfy

$$\begin{aligned} \left\{a^r_{\vec{p}}, a^{s\,\dagger}_{\vec{q}}\right\} = \left\{b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\right\} &= (2\pi)^3\delta^3(\vec{p}-\vec{q})\delta^{rs} \\ \left\{a^r_{\vec{p}}, a^s_{\vec{q}}\right\} = \left\{a^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\right\} = \left\{b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\right\} &= 0 \end{aligned}$$

and also their hermitean conjugates.

To see the nontrivial one, note

$$\begin{aligned}\left\{\psi(\vec{x},t),\psi^\dagger(\vec{y},t)\right\} &= \int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec p}E_{\vec q}}}\sum_{r,s}\\ &\quad\left(\left\{a^r_{\vec p},a^{s\,\dagger}_{\vec q}\right\}e^{-i(p^0-q^0)t}e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec y)}u^r(\vec p)u^{s\,\dagger}(\vec q)\right.\\ &\quad\left.+\left\{b^{r\,\dagger}_{\vec p},b^s_{\vec q}\right\}e^{i(p^0-q^0)t}e^{i(\vec p\cdot\vec x-\vec q\cdot\vec y)}v^r(\vec p)v^{s\,\dagger}(\vec q)\right)\\ &= \int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec p}E_{\vec q}}}\sum_{r,s}(2\pi)^3\delta^3(\vec p-\vec q)\delta^{rs}\\ &\quad\left(e^{-i(p^0-q^0)t}e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec y)}u^r(\vec p)u^{s\,\dagger}(\vec q)\right.\\ &\quad\left.+e^{i(p^0-q^0)t}e^{i(\vec p\cdot\vec x-\vec q\cdot\vec y)}v^r(\vec p)v^{s\,\dagger}(\vec q)\right)\\ &= \int\frac{d^3p}{(2\pi)^3}\frac{e^{-i\vec p\cdot(\vec x-\vec y)}}{2E_{\vec p}}\sum_r\left(u^r(\vec p)u^{r\,\dagger}(\vec p)+v^r(-\vec p)v^{r\,\dagger}(-\vec p)\right),\end{aligned}$$

where in the last term we replaced the integral over $\vec p$ by $-\vec p$. But from (3.66–3.67),

$$\sum_r u^r(\vec p)u^{r\,\dagger}(\vec p) = (\gamma\cdot p+m)\gamma^0 = E_{\vec p}-\vec\alpha\cdot\vec p+m\beta$$

and $\displaystyle\sum_r v^r(\vec p)v^{r\,\dagger}(\vec p) = (\gamma\cdot p-m)\gamma^0 = E_{\vec p}-\vec\alpha\cdot\vec p-m\beta,$

so $\displaystyle\sum_r\left(u^r(\vec p)u^{r\,\dagger}(\vec p)+v^r(-\vec p)v^{r\,\dagger}(-\vec p)\right) = 2E_{\vec p}\,\mathbb{I}.$

So $\displaystyle\left\{\psi_a(\vec x,t),\psi^\dagger_b(\vec y,t)\right\} = \int\frac{d^3p}{(2\pi)^3}e^{-i\vec p\cdot(\vec x-\vec y)}\mathbb{I}_{ab} = \delta^3(\vec x-\vec y)\delta_{ab}.$

We will work out the Hamiltonian

$$H = \int d^3x\bar\psi(-i\vec\gamma\cdot\vec\nabla+m)\psi$$

in terms of (1) and (2), noting that

$$\begin{aligned}(-i\vec\gamma\cdot\vec\nabla+m)\psi &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\sum_s\Big(a^s_{\vec p}\,(\vec\gamma\cdot\vec p+m)\,u^s(p)e^{-ip\cdot x}\\ &\qquad +b^{s\,\dagger}_{\vec p}\,(-\vec\gamma\cdot\vec p+m)\,v^s(p)e^{ip\cdot x}\Big)\end{aligned}$$

But as $(\gamma^\mu p_\mu-m)u(\vec p)=0=(\gamma^\mu p_\mu+m)v(\vec p)$, the parentheses in front of u and v reduce to $E_{\vec p}\gamma^0$ for u and $-E_{\vec p}\gamma^0$ for v. So

$$\begin{aligned}H &= \int d^3x\frac{d^3q}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec q}}}\sum_r\left(b^r_{\vec q}v^{r\,\dagger}(\vec q)e^{-iq\cdot x}+a^{r\,\dagger}_{\vec q}u^{r\,\dagger}(\vec q)e^{iq\cdot x}\right)\\ &\qquad\sqrt{2E_{\vec p}}\sum_s\left(a^s_{\vec p}u^s(\vec p)e^{-ip\cdot x}-b^{s\,\dagger}_{\vec p}v^s(\vec p)e^{ip\cdot x}\right)\\ &= \int d^3x\frac{d^3q}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac12\sqrt{\frac{E_{\vec p}}{E_{\vec q}}}\sum_r\sum_s\\ &\qquad\Big[b^r_{\vec q}a^s_{\vec p}v^{r\,\dagger}(\vec q)u^s(\vec p)e^{-i(p+q)\cdot x}-b^r_{\vec q}b^{s\,\dagger}_{\vec p}v^{r\,\dagger}(\vec q)v^s(\vec p)e^{i(p-q)\cdot x}\\ &\qquad +a^{r\,\dagger}_{\vec q}a^s_{\vec p}u^{r\,\dagger}(\vec q)u^s(\vec p)e^{-i(p-q)\cdot x}-a^{r\,\dagger}_{\vec q}b^{s\,\dagger}_{\vec p}u^{r\,\dagger}(\vec q)v^s(\vec p)e^{i(p+q)\cdot x}\Big]\\ &= \int\frac{d^3p}{(2\pi)^3}\frac12\sum_r\sum_s\Big[b^r_{-\vec p}a^s_{\vec p}v^{r\,\dagger}(-\vec p)u^s(\vec p)e^{-2iE_pt}-b^r_{\vec p}b^{s\,\dagger}_{\vec p}v^{r\,\dagger}(\vec p)v^s(\vec p)+\\ &\qquad a^{r\,\dagger}_{\vec p}a^s_{\vec p}u^{r\,\dagger}(\vec p)u^s(\vec p)-a^{r\,\dagger}_{-\vec p}b^{s\,\dagger}_{\vec p}u^{r\,\dagger}(-\vec p)v^s(\vec p)e^{2iE_pt}\Big]\\ &= \int\frac{d^3p}{(2\pi)^3}E_{\vec p}\sum_r\Big[-b^s_{\vec p}b^{s\,\dagger}_{\vec p}+a^{s\,\dagger}_{\vec p}a^s_{\vec p}\Big]\approx\int\frac{d^3p}{(2\pi)^3}E_{\vec p}\sum_r\Big[a^{s\,\dagger}_{\vec p}a^s_{\vec p}+b^{s\,\dagger}_{\vec p}b^s_{\vec p}\Big]\end{aligned}$$

where in the second line the x integral gives $(2\pi)^3\delta^3(\vec p\pm\vec q)$ which cancels the q integral, and in the third line we can use $u^{r\,\dagger}(-\vec p)\,v^s(\vec p)=v^{r\,\dagger}(-\vec p)\,u^s(\vec p)=0$ and $u^{r\,\dagger}(\vec p)\,u^s(\vec p)=v^{r\,\dagger}(\vec p)\,v^s(\vec p)=2E_{\vec p}\delta^{rs}$ from 3.60, 3.63 and 3.64, and we dropped an infinite constant from the anticommutator $\{b^{s\,\dagger}_{\vec p},b^s_{\vec p}\}$ in the last expression.

Notice that H has the same form as for a complex scalar field, and the commutation relations $\left[H,a^\dagger\right]$ and $\left[H,b^\dagger\right]$ work the same, even though the commutators of a and $a^\dagger$ are replaced with anticommutators, because H is quadratic, and

$$\sum_p\left[a^\dagger_{\vec p}a_{\vec p},a^\dagger_{\vec q}\right]=\sum_p a^\dagger_{\vec p}\left\{a_{\vec p},a^\dagger_{\vec q}\right\}-\sum_p\left\{a^\dagger_{\vec p},a^\dagger_{\vec q}\right\}a_{\vec p}=a^\dagger_{\vec q}.$$

Please read page 58 and the top of page 59. Then we can then skip to the bottom of page 60, as we have already discussed the Lorentz transformation properties. But do read the last paragraph on p. 60 and pp 61-65.