Generating Functions for Diagrams

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The partition function and the energy functional

Before we continue with the computation of the effective action, let's review the meanings of the generating functions we have already met. For simplicity we consider only a scalar field theory. We saw that the vacuum expectation value of the time ordered products of fields is given in terms of the functional integral

$$Z[J] = \int \mathcal{D}\phi \; e^{i\int d\mathbf{x}(\mathcal{L}(\phi) + J\phi)},$$

with

$$\mathcal{G}^n(x_1, \cdots, x_n) := \langle\Omega|\, T\phi(x_1)\ldots\phi(x_n)\,|\Omega\rangle = Z[0]^{-1}\left(\prod_{j=1}^{n} \frac{-i\delta}{\delta J(x_j)}\right) Z[J]\Bigg|_{J=0}.$$

We might have done better to define $W[J] = Z[0]^{-1}Z[J]$, with $W[0] = 1$, for the last equation shows that $W[J]$ is the generating function for for the V.E.V's, in the sense that $1/n!$ times the coefficient of $J(x_1)J(x_2)\ldots J(x_n)$ in $W[J]$ is the Green's function $\mathcal{G}^n(x_1, \cdots, x_n)$ of the theory. This function is given, in perturbation theory, by the sum of all diagrams with n external points and **no fully disconnected pieces**. It does contain pieces not connected to each other, as long as each piece has at least one external point. On the other hand, $Z[J]$ could be thought of as the sum over all graphs, including fully disconnected ones, though there is an ambiguous constant multiplying it.

The role as generating function might be clearer if we consider the external points on all diagrams to be associated with $\int d^D x J(x)$ rather than with 1, and then, as the vertices are now indistinguishable, the $1/n!$ factor can be considered a symmetry factor.

Let us define $G^n(x_1, \cdots, x_n)$ to be the sum of all (fully) connected diagrams with n external points, and define its generating function

$$iX[J] = \sum_n \frac{i^n}{n!}\left(\prod_{j=1}^{n}\int d^D x_j J(x_j)\right) G^n(x_1, \cdots, x_n).$$

I will show that $W[J] = e^{iX[J]}$.

For every graph in the n'th order term $\mathcal{G}^n(x_1, \cdots, x_n)$ in $W[J]$, there is some connected diagram attached to the point x_1 and some number $r-1$ of the other x's. Thus

$$\mathcal{G}^n(x_1, \cdots, x_n) = \sum_{r=1}^{n} \sum_{\{y_2 \ldots y_r\}} G^r(x_1, y_2, \ldots, y_r)\mathcal{G}^{n-r}(\text{other } x\text{'s}),$$

where the second sum is over all choices of $r-1$ y's from the $n-1$ x's (other than x_1).

If we multiply by $i^n J(x_2) \ldots J(x_n)/(n-1)!$ and integrate over $x_2 \ldots x_n$, each of the $\binom{n-1}{r-1}$ choices of the y_j from the x_k contribute equally. If we then sum over n, the left hand side becomes

$$\sum_{n=1}^{\infty} \frac{i^n}{(n-1)!} \prod_{j=2}^{n} \left(\int dx_j J(x_j) \right) \mathcal{G}^n(x_1, \cdots, x_n) = \frac{\delta}{\delta J(x_1)} W[J],$$

while the right hand side becomes

$$\sum_n \sum_{r=1}^{n} \frac{i^n}{(r-1)!(n-r)!} \prod_{j=2}^{r} \left(\int dx_j J(x_j) \right) G^r(x_1, \ldots, x_r)$$
$$\times \prod_{k=1}^{n-r} \left(\int dx_k J(x_k) \right) \mathcal{G}^{n-r}(x_1, \ldots, x_{n-r}).$$

With $m = n - r$ the sums can be restated as $\sum_{r=1}^{\infty} \sum_{m=0}^{\infty}$, and we get the simple product of

$$\sum_{r=1}^{n} \frac{i^r}{(r-1)!} \prod_{j=2}^{r} \left(\int dx_j J(x_j) \right) G^r(x_1, \ldots, x_r) = \frac{\delta}{\delta J(x_1)} iX[J]$$

with

$$\sum_m \frac{i^m}{(m)!} \prod_{k=1}^{m} \left(\int dx_k J(x_k) \right) \mathcal{G}^m(x_1, \ldots, x_{n-r}) = W[J].$$

Thus we have the differential equation

$$\frac{\delta}{\delta J(x_1)} W[J] = \left(\frac{\delta}{\delta J(x_1)} iX[J] \right) W[J],$$

which, together with the initial conditions $W[0] = 1$, $X[0] = 0$, gives us

$$W[J] = e^{iX[J]}.$$

Thus we see that, if we ignore the undetermined overall constant in $Z[J]$, or use $W[J]$ instead, the generating function for fully connected graphs is just $X[J] = E[J]$. My notation, W and X instead of Z and E, is used by Bailin and Love and some other references, but a lot of the notation seems not to be well fixed. Everyone seems to agree on $\Gamma[\phi_{\rm cl}]$.

The classical field and the effective action

We now turn to the classical field $\phi_{\rm cl}(x)$ and the effective action. The definition of the classical field is

$$\phi_{\rm cl}(x) = -\frac{\delta}{\delta J(x)} E[J].$$

We can consider the classical field to be a functional of the source, $\phi_{\rm cl}(x)[J]$. The effective action $\Gamma[\phi_{\rm cl}]$ is defined by be Legendre transform

$$\Gamma[\phi_{\rm cl}] = -E[J] - \int d^4x J(x) \phi_{\rm cl}(x),$$

from which we saw that

$$\frac{\delta \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(x)} = -J(x).$$

Note the implicit assumption that the functional $\phi_{\rm cl}(x)[J]$ can be inverted, and therefore Γ can be regarded as a functional of $\phi_{\rm cl}$ alone, depending only indirectly on J.