The beta function $B(x, y)$

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The Beta function is defined, for $\text{Re}\ \alpha > 0$, $\text{Re}\ \beta > 0$, as

$$B(\alpha, \beta) = \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1}.$$

To evaluate this, consider

$$\begin{aligned}
\Gamma(\alpha+\beta)B(\alpha,\beta) &= \int_0^\infty t^{\alpha+\beta-1}e^{-t}dt \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \\
&= \int_0^\infty t^{\beta-1}e^{-t}dt \int_0^t du\, u^{\alpha-1}(1-\frac{u}{t})^{\beta-1} \\
&= \int_0^\infty e^{-t}dt \int_0^t du\, u^{\alpha-1}(t-u)^{\beta-1} \\
&= \int_0^\infty du\, u^{\alpha-1} \int_u^\infty e^{-t}(t-u)^{\beta-1}dt \\
&= \int_0^\infty du\, u^{\alpha-1}e^{-u} \int_0^\infty e^{-v}v^{\beta-1}dv
\end{aligned}$$

$$= \Gamma(\alpha)\Gamma(\beta),$$

so

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Other interesting forms:
By writing $x = \sin^2\theta$ we have

$$B(\alpha,\beta) = 2\int_0^{\pi/2} (\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}d\theta.$$

By writing $x = 1/(1+u)$ we have

$$B(\alpha,\beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}\, du.$$

$\Gamma(z)$ has a pole of residue 1 at $z = 0$, but it is important to have the finite term as well. For infinitesimal ϵ we have

$$\Gamma(1+\epsilon) = \int_0^\infty t^\epsilon e^{-t}dt \approx \int_0^\infty (1+\epsilon\ln t)e^{-t}dt = 1 - \epsilon\gamma,$$

where

$$\gamma = -\int_0^\infty (\ln t)\, e^{-t} dt = 0.57721\,56649\ldots$$

is known as the Euler's constant or Mascheroni's constant and can also be written as

$$\gamma = \lim_{m\to\infty} \left(\sum_{j=1}^{m} \frac{1}{j} - \ln m \right).$$

Thus as $z \to 0$,

$$\Gamma(z) = z^{-1}\Gamma(1+z) = \frac{1}{z} - \gamma + \mathcal{O}(z).$$

There is lots more about Gamma functions in Whittaker and Watson, Modern Analysis, chapter 12.