

Functional Integral for a Scalar Field.

We have seen that the time evolution operator in quantum mechanics can be understood as a functional integral

$$\langle q_f, t_f | e^{-iH[t_f-t_i]} | q_i, t_i \rangle = \int \mathcal{D}q \mathcal{D}p \exp i \int dt [p\dot{q} - H(q, p, t)],$$

where from now on $\hbar = 1$. This is quite general, so we are not restricted to a finite number of degrees of freedom q . In particular, we could consider a scalar field with Hamiltonian

$$H = \int d^3\vec{x} \left\{ \frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2} (\nabla\phi)^2(\vec{x}) + V(\phi(\vec{x})) \right\}.$$

[For a free scalar field of mass m , $V(\phi) = \frac{1}{2}m^2\phi^2$.] Then we need

$$\int \mathcal{D}\phi \mathcal{D}\pi \exp i \int d^3\vec{x} dt \left[-\frac{1}{2} \pi^2(x^\mu) + \pi(x^\mu) \dot{\phi}(x^\mu) \right] \exp i \int d^3\vec{x} dt \left[-\frac{1}{2} (\nabla\phi)^2 - V(\phi) \right].$$

Only the first exponential depends on $\pi(x^\mu)$, and we can treat each point of space-time independently. Using

$$\int \frac{dx}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} \lambda x^2 + Bx \right] = \frac{e^{B^2/2\lambda}}{\sqrt{\lambda}},$$

with $\lambda = i\Delta^4 x$, and ignoring the field-independent normalization factors, we have

$$\int \mathcal{D}\pi \exp i \int d^3\vec{x} dt \left[-\frac{1}{2} \pi^2(x^\mu) + \pi(x^\mu) \dot{\phi}(x^\mu) \right] \sim \exp i \int d^3x dt \frac{1}{2} \dot{\phi}^2,$$

and thus the functional integral becomes

$$\int \mathcal{D}\phi \exp i \int d^4x \left[\frac{1}{2} \dot{\phi}^2(x^\mu) - \frac{1}{2} (\nabla\phi)^2 - V(\phi) \right] = \int \mathcal{D}\phi e^{i \int d^4x \mathcal{L}(\phi)},$$

where the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi).$$

We now have a form of quantum mechanics based on the Lagrangian rather than the Hamiltonian, and except for the integration region (which goes from t_i to t_f in time but includes all of space) our formulation is Lorentz invariant. It will also have symmetries under any transformation which leaves the Lagrangian density \mathcal{L} invariant.

Correlation Functions

In quantum mechanics, the expectation values or transitions of operators are given by $\langle \psi_f | \mathcal{O}_1 \mathcal{O}_2 | \psi_i \rangle$, where we generally think of operators at a fixed time. But in field theory we need to think of operators acting at arbitrary points in space-time. In the Schrödinger picture, where the fundamental operators are considered time-independent but the states evolve, the evolution $\phi_i \rightarrow \phi_f$ under the influence of some operators $\mathcal{O}'(x_2^\mu) \mathcal{O}(x_1^\mu)$ is given by

$$\langle \phi_f | e^{-i(t_f-x_2^0)H} \mathcal{O}'_S(x_2^\mu) e^{-i(x_2^0-x_1^0)H} \mathcal{O}_S(x_1^\mu) e^{-i(x_1^0-t_i)H} | \phi_i \rangle,$$

where I have assumed x_2 is later than x_1 . We may rewrite this in the Heisenberg picture, in which operators evolve while the states are fixed. The connection is

$$\mathcal{O}_H(\vec{x}, t) = e^{iHt} \mathcal{O}_S(\vec{x}) e^{-iHt}.$$

Then the transition amplitude is

$$\langle \phi_f | e^{-it_f H} \mathcal{O}'_H(x_2^\mu) \mathcal{O}_H(x_1^\mu) e^{it_i H} | \phi_i \rangle.$$

Consider the operators to be the fields themselves. Returning to the Schrödinger picture, we see that

$$\begin{aligned} & \langle \phi_f | e^{-it_f H} \phi_H(x_2^\mu) \phi_H(x_1^\mu) e^{it_i H} | \phi_i \rangle \\ &= \langle \phi_f | e^{-i(t_f-x_2^0)H} \phi_S(x_2^\mu) e^{-i(x_2^0-x_1^0)H} \phi_S(x_1^\mu) e^{-i(x_1^0-t_i)H} | \phi_i \rangle \\ &= \int_{t_2} \mathcal{D}\phi_2(\vec{x}) U(\phi_f, \phi_2, t_f, t_2) \phi_2(x_2^\mu) \int_{t_1} \mathcal{D}\phi_1(\vec{x}) U(\phi_2, \phi_1, t_2, t_1) \phi_1(x_1^\mu) U(\phi_1, \phi_i, t_1, t_i) \\ &= \int \mathcal{D}\phi \phi(x_2^\mu) \phi(x_1^\mu) \exp i \int_{t_i}^{t_f} d^4x \mathcal{L}(\phi). \end{aligned}$$

Note that the $\int \mathcal{D}\phi$ in the next-to-last line are integrals only over states at a given time, while the $\int \mathcal{D}\phi$ in the last line is a functional integral over space-time.

The connection of the operator expectation value to the functional integral assumed $x_2^0 > x_1^0$. The functional integral itself is unchanged under $x_1 \leftrightarrow x_2$, as the ϕ 's there are commuting c -numbers. But in the operator form $\phi_H(x_2^\mu)$ and $\phi_H(x_1^\mu)$ need not commute, and our expression is only correct if the later one occurs first. We therefore introduce the ‘‘Time ordering operator’’ T , which tells us to order any operators appearing in its scope in decreasing order of their times. This is not an operator on the Hilbert space;

rather it is a metaoperator which acts on our symbolic expressions, reordering them before they are allowed to physically act on the Hilbert space. Our corrected expression is thus

$$\langle \phi_f | e^{-it_f H} T(\phi_H(x_2^\mu) \phi_H(x_1^\mu)) e^{it_i H} | \phi_i \rangle = \int \mathcal{D}\phi \phi(x_2^\mu) \phi(x_1^\mu) \exp i \int_{t_i}^{t_f} d^4 x \mathcal{L}(\phi).$$

We are interested in evaluating correlation functions, which are the expected values of operator products in the vacuum, or lowest energy state. If we insert a complete set of energy eigenstates $\sum_n |n\rangle \langle n| = 1$ at the ends of our operator, we have

$$\begin{aligned} & \int \mathcal{D}\phi \phi_2(x_2^\mu) \phi_1(x_1^\mu) \exp i \int_{t_i}^{t_f} d^4 x \mathcal{L}(\phi) \\ &= \sum_n \sum_m \langle \phi_f | n \rangle \langle n | e^{-it_f E_n} T(\phi_H(x_2^\mu) \phi_H(x_1^\mu)) e^{it_i E_m} | m \rangle \langle m | \phi_i \rangle. \end{aligned}$$

Now we can formally extract the vacuum matrix elements by formally letting t_f approach $+\infty$ with a negative imaginary part, $t_f \rightarrow \infty(1 - i\epsilon)$. If we call the lowest state $\langle \Omega |$ and assume its energy is 0, all other states get a contribution $\exp -(E_n \epsilon \cdot \infty) \rightarrow 0$, and we extract only the vacuum state from the sum. Similarly we let $t_i \rightarrow -\infty(1 - i\epsilon)$. Assume the states $\langle \phi_f |$ and $\langle \phi_i |$ have some overlap with the vacuum state $\langle \Omega |$. Then

$$\begin{aligned} \langle \Omega | T \phi_H(x_2) \phi_H(x_1) | \Omega \rangle &= \frac{\langle \Omega | T \phi_H(x_2) \phi_H(x_1) | \Omega \rangle}{\langle \Omega | \Omega \rangle} \\ &= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \phi_2(x_2^\mu) \phi_1(x_1^\mu) \exp i \int_{-T}^T d^4 x \mathcal{L}(\phi)}{\int \mathcal{D}\phi \exp i \int_{-T}^T d^4 x \mathcal{L}(\phi)}. \end{aligned}$$