Physics 615 Fall, 2013

Homework #1 Due: Sept. 13 at 3:00 P.M.

In these problems, treat the fields, lagrangians and equations of motion classically.

1 Consider the Lagrangian for two free real scalar fields ϕ_j with equal masses, with

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{2} \partial_\mu \phi_j \partial^\mu \phi_j - \frac{1}{2} m^2 \sum_{j=1}^{2} \phi_j^2. \tag{1}$$

Define ϕ without an index to be the complex field

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}},$$

and rewrite the Lagrangian in terms of ϕ and $\phi^\dagger$, its complex conjugate. [Generally we use † for Hermitean conjugate, which is more appropriate if we consider ϕ as an operator, but here Hermitean and complex conjugate are synonymous.] Find the equations of motion for ϕ_j, and show that one gets the same results if we treat $\mathcal{L}(\phi, \phi^\dagger)$ as if ϕ and $\phi^\dagger$ were independent real fields. Also find the Hamiltonian density both in terms of ϕ_i and by treating ϕ and $\phi^\dagger$ as if they were independent, and compare.

2 Maxwell's theory of electromagnetism can be expressed in terms of a 4-vector field A_μ, with a Lagrangian density given by

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where the field strength tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and with the identification with usual Maxwell fields given by $E^i = F^{i0}$, $\epsilon_{ijk} B^k = -F^{ij}$. Note that the degrees of freedom are A_μ, not $F_{\mu\nu}$.

(a) Derive the equations of motion for A_μ, and reexpress them in terms of the electric and magnetic fields.

(b) Find the canonical momenta Π_μ conjugate to A^μ, and give their constitutive equations $\Pi_\mu = \Pi_\mu(A^\nu, \dot{A}^\nu)$ and reexpress in terms of electromagnetic fields. Can these be inverted to solve for $\dot{A}^\mu$ as functions of A^ν and Π^ν?
[Note: Π^μ are not the generalizations π^μ which we defined with the $\bar{\delta}$ for each field. Here the index μ refers to which field, A^μ, we are finding the conjugate to.]

3 The last Problem described a free electromagnetic field in the absence of charges and currents. To include such sources, add a term $-A_\mu j^\mu$ to the lagranian density, so

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu,$$

where $j^0 = \rho$ is the charge density, and $j^i = \vec{j}$ is the current density, in Heaviside units.

(a) Derive the new equations of motion for A_μ, and reexpress them in terms of the electric and magnetic fields.

(b) Show that the equations of motion are only consistent if the source satifies

$$\partial_\mu j^\mu = 0.$$