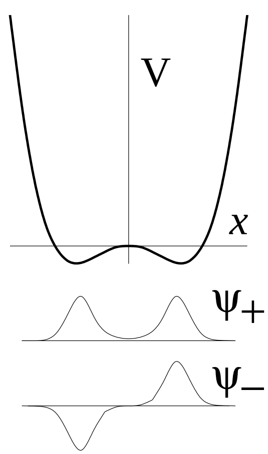
Physics 613 Lecture 22 April 22, 2014

Spontaneous Symmetry Breaking

We argued from Noether's theorem that if the physical laws have a continuous symmetry group, there existed for each symmetry generator a conserved current and conserved quantity, making only the reasonable assumptions that the vacuum state is invariant under the symmetry and that we could describe all states of the theory in terms of localized excitations, so that we could assume that the conserved current vanished at infinity. But today we will explore the loopholes in this theorem, in particular, spontaneous symmetry breaking.

We will first discuss a simple quantum mechanical problem, the double well in one dimension, with a potential given by $V = -ax^2 + bx^4$, where a and b are positive. For small x the potential is negative and will have a minimum at $x = \pm\sqrt{a/2b}$. Classically the lowest energy states would sit at either of those two points, and the state would not be symmetric under the symmetry $x \leftrightarrow -x$. Quantum mechanically, if the well is deep enough, one could approximate a low energy state by a harmonic oscillator ground state centered at one of these classical minima, either one, and we would have a doubly degenerate ground state. There would be negligible overlap between the wave function ψ_R describing a particle in the well at $x = \sqrt{a/2b}$ and the one, ψ_L at $x = -\sqrt{a/2b}$. But this is not exact — as long as the bump in the middle is finite, there is a finite overlap between ψ_R and ψ_L, and in fact the true energy eigenfunctions are $\psi_\pm \propto \psi_R \pm \psi_L$, with ψ_+ having a slightly lower energy than ψ_-, and therefore is the true ground state of the system, duly symmetric under the symmetry.



Although the wave function limited to positive x would not be an exact wave function, if the bump is high enough and the overlap small, the time it would take for tunneling to the true ground state might be very long.

Now consider what would happen if the variable x here were not a spatial index but some internal variable ϕ, and if there were eight such systems laid out in space at the corners of a cube. In addition to to the potential $V = \sum_j(-a)\phi_j^2 + b\left(\sum_j \phi_j^2\right)^2$, there were also a weak coupling of nearest neighbors $-c\sum \phi_j\phi_k$. We would still have an overall symmetry of changing the signs of all the ϕ_j's together, but there would be a ferromagnetic type coupling favoring the neighboring $\phi's$ lining up. There would still be an almost exact lowest energy state with all ϕ's in the positive well and another will all in the negative well, and now the tunneling would be very much longer, as the small overlap would be raised to the eighth power. So transitions between the two ground states would be nearly negligible.

Finally, let there be an infinite number of such dynamical variables, with some connection that prefers the dynamical variables be lined up. Now there is no overlap whatsoever, and the two ground states become true ground states. There is no local operator that can get you from one ground state to the other. The symmetry of changing the sign of all the variables is still a symmetry of the lagrangian, or of the equations of motion, but as the two states are completely independent, it makes no sense to talk about this symmetric situation. Instead, the state of a system is in the Hilbert space based on the excitations from one of the two vacuum states, and the symmetry is lost in the spectrum of states accessible.

These were not a continuous symmetries, but we can take a similar quantum-mechanical problem in two dimensions. Consider a single particle but rotating the potential about the z axis, replacing ϕ^2 by $\phi_1^2 + \phi_2^2$, so we have a particle rolling in the bottom of a wine bottle. A classical lowest energy state would have the particle at rest at $\rho = \sqrt{a/2b}$, at some arbitrary angle in the ϕ_1-ϕ_2 plane. But again, the quantum-mechanical ground state would have a wave function $\psi(|\phi|)$ independent of angle and therefore invariant under rotatation in that plane. If we had an infinite number of such systems, then there would be no overlap between states built on ground states with $\vec\phi$ centered at different angles.

Field Theory

Does this discussion seem artificial? It is just what we have in a field theory, where there are degrees of freedom at each point in space, and the kinetic energy term $(\vec\nabla\phi)^2$ term provides the ferromagnetic interaction.

Consider a theory with a multiplet of N real scalar fields[1], with[2]

$$\hat{\mathcal{L}} = \sum_{j=1}^{N}(\partial_\nu\phi_j)(\partial^\nu\phi_j) + \frac{1}{2}\mu^2\sum_{j=1}^{N}\phi_j^2 - \frac{\lambda}{4}\left(\sum_{j=1}^{N}\phi_j^2\right)^2 .$$

This lagrangian is, of course, invariant under SO(N), rotations of ϕ in the N-dimensional internal space. $\phi_j(x) \to O_{jk}\phi_k(x)$, with O an orthogonal matrix, which leaves $\sum\phi^2$ invariant. Notice that the mass term has the wrong sign. Had that sign been minus, with a + in the potential energy, the potential's absolute minimum would have been at $\vec{\phi} = 0$, and the rotational symmetry would be intact. But with the plus in the lagrangian, $\vec{\phi} = 0$ is a local maximum of the potential energy, and not a classical ground state. Instead a classical lowest energy state will have $\vec{\phi} = \vec{\phi}_0$ with $\phi_0 := |\vec{\phi}_0| = \mu/\sqrt{\lambda}$. To get a state of lowest energy, we need not only that $V(\phi)$ is minimized at each point x, but also that the kinetic energy $(\vec{\nabla}\phi)^2$ is minimized, which means vanishing. So although the potential only tells us that $\vec{\phi}(x)$ should lie on a sphere of radius ϕ_0, minimizing the energy means it has to be the same $\vec{\phi}$ throughout space.

If we apply a global SO(N) transformation, we get a new state which is of the same energy as our state $\vec{\phi}_0$, but which has an overlap with $\vec{\phi}_0(x)$ at every point which is less that one, so that raised to the infinite power from the infinite number of points $\vec{x}$, there is zero overlap! A system in one of these ground states can never get to another, equivalent, ground state.

In all our field-theoretic considerations so far, we have assumed the vacuum arises somehow from the classical state where the fields are all zero. We know there are vacuum fluctuations, but they are fluctuations about the $\phi = 0$ state. But if there is a classical state of lower energy (or, more accurately, energy density), we should expect our vacuum and our low excitations from it to be based on this lowest-energy state, not the $\phi = 0$ state.

How to proceed? We can rewrite our fields $\vec{\phi}(x) = \vec{\phi}_0 + \vec{\eta}(x)$. We can choose our $\vec{\phi}_0$ to be anywhere on the minimal surface, so let us choose it in the N direction, $\phi_0 = (0, \cdots, 0, \mu/\sqrt{\lambda})$. As $\vec{\phi}_0$ is a constant, the kinetic energy term in $\mathcal{L}$ is just $\sum_{j=1}^{N}(\partial_\nu\eta_j)(\partial^\nu\eta_j)$. The potential energy as a function of η is now

$$V(\eta) = -\frac{\mu^2}{2}\left(\sum_{j=1}^{N-1}\eta_j^2 + (\eta_N + \mu/\sqrt{\lambda})^2\right) + \frac{\lambda}{4}\left(\sum_{j=1}^{N-1}\eta_j^2 + (\eta_N + \mu/\sqrt{\lambda})^2\right)^2 .$$

As η_N is now being treated differently, let's call it σ. We have

$$\begin{aligned} V(\eta) &= -\frac{\mu^4}{2\lambda} - \frac{\mu^3}{\sqrt{\lambda}}\sigma - \frac{\mu^2}{2}\sigma^2 - \frac{\mu^2}{2}\eta_j^2 + \frac{\lambda}{4}\left(\eta_j^2 + \sigma^2 + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \frac{\mu^2}{\lambda}\right)^2 \\ &= -\frac{\mu^4}{4\lambda} + \left(-\frac{\mu^3}{\sqrt{\lambda}} + \frac{\mu^3}{\sqrt{\lambda}}\right)\sigma + \left(-\frac{1}{2} + 1 + \frac{1}{2}\right)\mu^2\sigma^2 + (-1+1)\frac{\mu^2}{2}\eta_j^2 \\ &\quad +\mu\sqrt{\lambda}\,\sigma^3 + \mu\sqrt{\lambda}\,\sigma\eta_j^2 + \frac{\lambda}{4}\left(\eta_j^2\right)^2 + \frac{\lambda}{4}\sigma^4 + \frac{\lambda}{2}\sigma^2\eta_j^2 \end{aligned}$$

where the sums η_j now have $j = 1..N-1$, with $\eta_j^2 := \sum_{j=1}^{N-1}\eta_j^2$.

Notice the linear term in σ vanishes, as it must, because the minimum is at $\sigma = 0$, $\eta_j = 0$. Notice also that the η_j has lost its quadratic term, so these $N-1$ degrees of freedom have become massless. Finally, notice that the σ has developed a mass $\sqrt{2}\,\mu$ with the correct positive sign in the potential. Our lagrangian has now become

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\nu\eta_j)^2 + \frac{1}{2}(\partial_\nu\sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\,\mu\sigma^3 - \sqrt{\lambda}\,\mu\sigma(\eta_j)^2 \\ &\quad -\frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\sigma^2\eta_j^2 - \frac{\lambda}{4}(\eta_j^2)^2. \end{aligned}$$

We have quartic terms for all the fields, with the correct signs to keep energy bounded from below, though we now have cubic interactions, of the σ with the η's and with itself. The theory still has a symmetry under rotations in the $N-1$ dimensional space $j = 1..N-1$, but it has lost symmetry under rotations which include the N'th dimension. It also does not have symmetry under $\sigma \leftrightarrow -\sigma$.

The model we have just considered is called the linear sigma model. With $N = 4$, we are left with 3 massless fields. This was taken in the '60's as a model describing the isotriplet of pi mesons, which are light compared to all other hadrons. As the pions are made of the u-d isodoublet of quarks and their antiquarks, if they are massless, as we discussed briefly in lecture 19, there could be a chiral symmetry with conserved vector and axial vector

[1]The book, in section 17.5, treats a single complex field, but that is a doublet of real scalars.

[2]I was tempted to consider a more general $V(\vec{\phi})$, where V is invariant under some Lie subgroup G of O(N). We need the symmetry to be $\subset O(N)$ because the kinetic term needs to be invariant. What we say here applies also to this more general case, but it is easiest if we restrict our discussion to rotations and ϕ^4 theory.

currents and conserved charges $\hat{Q}_j$ and $\hat{Q}_{j,5}$ which form an SO(4) symmetry. As we showed there, this has a Lie algebra equivalent to $SU(2) \times SU(2)$, with generators $\hat{Q}_{j,R}$ and $\hat{Q}_{j,L}$. If we imagine that this symmetry is somehow spontaneously broken, and in addition there is a small explicit breaking, it might explain both the near masslessness of the pions and also the correction to that, connected to the not-quite-conservation of the axial vector current in weak interactions.

More General Symmetries

Let us consider now that the symmetry group is some general Lie group $\mathcal{G}$, with fields that transform linearly under infinitesimal transformations generated by its Lie algebra $\mathfrak{G}$. As we did for SO(N), we consider that the lowest energy state might not be invariant under the full symmetry group $\mathcal{G}$ of the Lagrangian. Then we must choose the ground state, or vacuum, to be one lowest energy state, and there will be some of the transformations in $\mathcal{G}$ that do not leave the ground state invariant. There will be a subgroup $\mathcal{K}$ which does leave the vacuum state invariant. We will discuss Goldstone's theorem, which states that for each such generator in $\mathfrak{G}/\mathfrak{K}$, there will be a massless particle. This is fancy language to say this: The generators are a basis of the Lie algebra of infinitesimal symmetry transformation, which is a vector space. So we make sure to choose a basis so that some of the elements leave the vacuum invariant and the others are perpendicular to those. The ones that leave the vacuum invariant continue to generate a (smaller) symmetry of the theory, but the others each generate a massless boson. Now let's prove that.

The lagrangian is a sum of a kinetic term which involves space-time derivatives of the fields, and a potential term $V(\phi)$ which doesn't. The vacuum state is at $\phi = \phi_0$, which is a minimum of V, so we can expand V in a Taylor series

$$V(\phi) = V(\phi_0) + \frac{1}{2}\sum_{jk}(\phi-\phi_0)_j(\phi-\phi_0)_k \left(\frac{\partial^2 V}{\partial\phi_j\,\partial\phi_k}\right)_{\phi_0} + \cdots,$$

where there is no linear term because at the minimum all first derivatives must vanish. Let us define

$$m^2_{jk} = \left(\frac{\partial^2 V}{\partial\phi_j\,\partial\phi_k}\right)_{\phi_0}.$$

This is a symmetric matrix which is positive semidefinite, because ϕ_0 is a minimum, not just a saddle point.

Now consider an infinitesimal global symmetry transformation which takes the fields

$$\phi_j \to \phi_j + \alpha\Delta_j(\phi).$$

The change can depend on all the fields. As this is a symmetry transformation, it leaves V invariant,

$$V(\phi_j) = V(\phi_j + \alpha\Delta_j(\phi)) \qquad \text{so} \qquad \Delta_j(\phi)\frac{\partial V(\phi)}{\partial\phi_j} = 0.$$

Differentiate this with respect to ϕ_k and go to the vacuum value,

$$0 = \sum_j \left(\frac{\partial\Delta_j(\phi)}{\partial\phi_k}\right)_{\phi_0}\left(\frac{\partial V(\phi)}{\partial\phi_j}\right)_{\phi_0} + \sum_j \Delta_j(\phi_0)\left(\frac{\partial^2 V(\phi)}{\partial\phi_j\,\partial\phi_k}\right)_{\phi_0}.$$

But the second factor in the first term vanishes, so we have

$$\sum_j m^2_{kj}\Delta_j(\phi_0) = 0.$$

We see that if the vector $\Delta_j(\phi_0)$ is not zero, it is an eigenvector of the mass-squared matrix with eigenvalue zero. That is, it is a massless particle, called a Goldstone boson. On the other hand, the generators which leave the vacuum invariant have $\Delta_j(\phi_0) = 0$, so these do not correspond to Goldstone bosons.

Next time we will use these Goldstone bosons to avoid the theorem we had about the Ward identity protecting the massless gauge fields from acquiring masses.