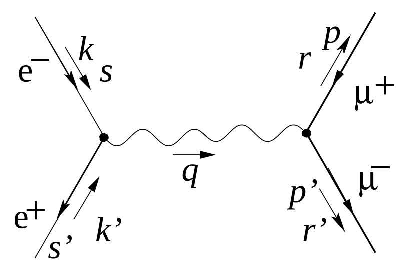
Physics 613 Lecture 14 March 25, 2014

$e^+ e^- \to X \bar{X}$, Inclusive Cross Sections

$e^+ e^- \to X \bar{X}$

From the Feynman rules, we see that the diagram describing $e^- \mu^-$ scattering looks just like the diagram for $e^+ e^- \to \mu^- \mu^+$, just turned on its side, with some momentum arrows reversed. In fact, for any fermion X, the invariant matrix element for this process is



$$i\mathcal{M} = -i\frac{eQ}{q^2}\,\bar{u}_X(\vec{p}\,', r')\gamma^\mu v_X(\vec{p}, r) \times \bar{v}_e(\vec{k}\,', s')\gamma_\mu u_e(\vec{k}, s)$$

When we evaluate $\sum |\mathcal{M}|^2$ summed over spins, we get e^2Q^2/q^4 times the product of

$$\begin{aligned} M^{\mu\rho} &= \frac{1}{2}\bar{u}_X(\vec{p}\,', r')\gamma^\mu v_X(\vec{p}, r)\bar{v}_X(\vec{p}, r)\gamma^\rho u_X(\vec{p}\,', r') = \frac{1}{2}\mathrm{Tr}[(\not{p}'+M)\gamma^\mu(\not{p}-M)\gamma^\rho] \\ &= 2p'^\mu p^\nu + 2p'^\nu p^\mu - 2p'\cdot p g^{\mu\nu} - 2M^2 g^{\mu\nu} = 2p'^\mu p^\nu + 2p'^\nu p^\mu - q^2 g^{\mu\nu}, \end{aligned}$$

with

$$L_{\mu\rho} = \frac{1}{2}\mathrm{Tr}[(\not{k}' + m)\gamma_\mu(\not{k} - m)\gamma_\rho] = 2k'_\mu k_\nu + 2k'_\nu k^\mu - q^2 g_{\mu\nu}.$$

Note that each of L and M are exactly the negative of what we got in considering $e^- \mu^-$ scattering, with the signs of k' and p changed because the momentum arrows on those lines have been reversed. The product which determines the unpolarized scattering cross section is exactly the same function,

$$\frac{1}{4}\sum_{r,r',s,s'}\mathcal{M}\mathcal{M}^* = \frac{4e^2Q^2}{q^4}\left[2p'\cdot k'p\cdot k + 2p'\cdot kp\cdot k' + (m^2 + M^2)q^2\right]$$

as we had before, though it would be a more generalizable statement if we said the function of the momenta is the same if we consider all momenta defined with the arrows in the same direction for the two processes. Then the amplitude changes sign for each reversed particle, and the square that determines the cross section is exactly the same function. We note that for this conversion the k' and p changed signs. This is consistent with the $q = p' - p$ of $e\,\mu$ scattering being converted into $q = p' + p$ here. Of course the kinematics is different — now $q^2 = s$, and, in the center of mass,

$$p \cdot k = p' \cdot k' = E^2 + |\vec{p}||\vec{k}| \cos\theta, \qquad p \cdot k' = p' \cdot k = E^2 - |\vec{p}||\vec{k}| \cos\theta.$$

If we neglect the electron mass, we get

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{|\vec{p}|}{256\pi^2 E\, s} \sum |\mathcal{M}|^2 \\
&= \frac{e^2 Q^2}{64\pi^2 s^3} \sqrt{1 - \frac{M^2}{E^2}}\, 16 \left[E^4(1 + \cos^2\theta) + E^2 M^2 (1 - \cos^2\theta)\right] \\
&= \frac{Z^2 \alpha^2}{4s} \sqrt{1 - \frac{M^2}{E^2}} \left[\left(1 + \frac{M^2}{E^2}\right) + \left(1 - \frac{M^2}{E^2}\right)\cos^2\theta\right],
\end{aligned}$$

where $Q = Ze$. Integrating over solid angle, we get

$$\sigma_{\text{total}} = \frac{4\pi Z^2 \alpha^2}{3s} \sqrt{1 - \frac{M^2}{E^2}} \left(1 + \frac{1}{2}\frac{M^2}{E^2}\right) \underset{s \gg M^2}{\longrightarrow} Z^2 \frac{4\pi\alpha^2}{3s}.$$

This cross section has been well verified for $\mu^- \mu^+$ and for $\tau^- \tau^+$.

Crossing Symmetry

I will come back to say more about $e^+ e^- \to X \bar{X}$ in a moment, but first I want to say more about the relationship of invariant amplitudes for processes which differ by having an external particle shifted from incoming particle to outgoing antiparticle, or vice versa. In the Feynman rules, the only differences are the direction of the momentum and switching $u \leftrightarrow v$. If we sum over spins, this switch just changes the sign of the mass associated with the momentum, so $\not{p} + m \to -\not{p} - m$, if we call the momentum of the antiparticle $-p$. Things are a little more awkward if we try to identify spins, because then we would need to relate the ϕ^r and χ^s, which we have left vague, but even with spins we can say that the appropriate amplitudes are the same except for a sign change, and an analytic continuation in the variables. For example, here we have $q^2 = (k + k')^2 > 0$, while in μe scattering we had $q^2 = (k - k')^2 < 0$, but $\sum |\mathcal{M}|^2$ is the same *function* of q^2, even though it is evaluated for different values of q^2 in the two processes. This is a general feature of the invariant scattering amplitudes.

R

For leptons, which we can treat as elementary Dirac particles, we can calculate the scattering amplitudes explicitly. If there were nothing but leptons, we could say that the total cross section for $e^+ \, e^-$ scattering would be

$$\sigma_{\rm tot} = \sigma_{e^+ \, e^- \to e^+ \, e^-} + \sigma_{e^+ \, e^- \to \mu^+ \, \mu^-} + \sigma_{e^+ \, e^- \to \tau^+ \, \tau^-}$$

plus some much smaller contribution with 4 leptons or more in the final state. But much more significant is the production of hadrons, strongly interacting particles. In the book the process $e^+ \, e^- \to \pi^+ \, \pi^-$ is considered, treating the pions as pseudoscalar point particles but then using form factors to handle their unknown structure. We do not have a fundamental theory which enables us to calculate structure of hadrons like the pion, even though we believe it is determined by quantum chromodynamics, QCD, because the interactions between hadrons are so strong. But it turns out that at high energies, these interactions between quarks and gluons are weakened.

Now if there were no strong interactions between quarks, each species of quark could be pair produced in $e^+ \, e^-$ scattering. The cross section for a quark of type j, at energies much greater than the quark mass, would just be

$$\sigma_{e^+ \, e^- \to q_j \, \bar{q}_j} = Z_j^2 \frac{4\pi\alpha^2}{3s} = Z_j^2 \, \sigma_{e^+ e^- \to \mu^+ \mu^-} .$$

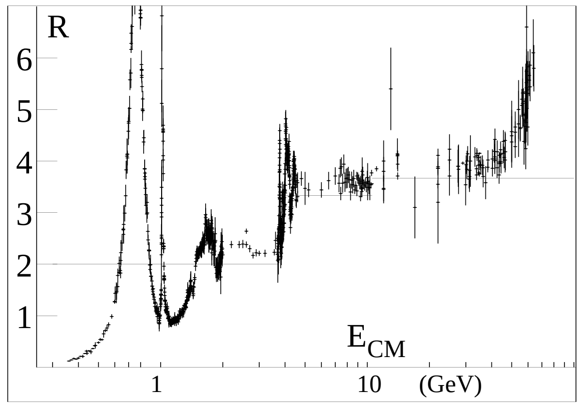
Here Z is[1] the ratio of the quark charge to the positron's charge, and is 2/3 for the up quark[2] and $-1/3$ for the down and strange quarks[3]. If we define

[1] Z is not the usual name, which the book calls e_a and many call Q, but why not use the nuclear physics definition?

[2] and also the charm and top quarks.

[3] and bottom quark.

$$R(s) = \frac{\sigma_{e^+\,e^-\to\text{all hadrons}}}{\sigma_{e^+\,e^-\to\mu^+\,\mu^-}},$$



we would expect $R = \sum_j Z_j^2$, where the sum should include all quarks light enough to be pair produced. Actually there are hadronic resonance states R which, like the Δ^{++} in $\pi^+ p$ scattering, produce strong peaks when $s = M_R^2$, and there are also threshold effects $\sim |\vec{k}|/E$ when the final particles are not ultrarelativistic. Also, as quarks can emerge only coupled with other quarks or antiquarks, the thresholds are at twice the mass of the lightest meson containing the j'th quark, rather than at twice the "bare" mass of the undressed quark.

In the plot of R vs. E_{CM}, below 2 GeV we see the strong resonance peaks from the ρ^0 and ϕ^0 resonances, and then perhaps the threshold of having all three light quarks contributing. As the charges of u, d and s are 2/3, $-1/3$ and $-1/3$ respectively, we might have expected $R = (2/3)^2 + (-1/3)^2 + (-1/3)^2 = 2/3$. After twice the D-meson mass (2×1.87 GeV) we might expect an additional $(2/3)^2$ to bring R up to 10/9. But we see that the actual R does have that shape but at three times the value, $R \approx 2$ below 4 GeV, jumping to $R \approx 3\ 1/3$ from 4 up to η_b meson at 9.4 GeV, when the bottom quark starts contibuting its $3 \times (-1/3)^2$.

This is strong evidence that in fact each of these flavored quarks comes in three colors.

Inelastic $e - p$ Scattering

Earlier we discussed $e^- p$ elastic scattering, and we found that the answers could be expressed in terms of two form factors which are functions of q^2, where q^μ is the momentum transferred from the electron. All we could say about these form factors from first principles was that $\mathcal{F}_1(0) = 1$. To understand the q^2 dependence would necessitate understanding the strong interactions among the constituents of the proton. One feature of strong interaction scattering at high energies is that many particles are produced, mostly in the direction of the incident beams, with no particle having a large transverse momentum. Thus if an electron does transfer significant q^2 to part of a proton, it is unlikely to be shared with all the rest of the proton, the proton will come apart and we have inelastic, rather than elastic, scattering.

Now consider such scattering, $e^- p \to e^- + X$, where X is a multiparticle state with overall charge $+e$, baryon number $+1$, and total momentum p'_μ. Let k_μ and k'_μ be the initial and final electron momenta, and p_μ the initial proton momentum. As $q = k' - k$, $q^2 < 0$, so often people define $Q^2 = -q^2 > 0$. As we said, at high momentum transfer Q^2, the final hadronic state X is likely to consist of many particles, with an invariant mass squared $W^2 = p'^2$ considerably greater than the proton's M^2. It is still a good approximation to assume the electromagnetic interaction can be treated as a single photon exchange, so

$$\mathcal{M} = -\frac{e\bar{u}_e(\vec{k}', s')\gamma_\mu u_e(\vec{k}, s)}{q^2} \langle X| J^\mu |p, s\rangle .$$

In calculating the cross section, the electron part of $\sum |\mathcal{M}|^2$ will be unchanged from the elastic case, $L_{\mu\nu} = 2k'_\mu k_\nu + 2k'_\nu k^\mu + q^2 g_{\mu\nu}$, and the hadronic piece, which we will now call $W^{\mu\nu}$ is defined as

$$e^2 W^{\mu\nu} = \frac{1}{4\pi M}\frac{1}{2}\sum_s \sum_X (2\pi)^4 \delta^4(p' - p - q) \langle p, s| J^\mu(0) |X, p'\rangle \langle X, p'| J^\nu(0) |p, s\rangle .$$

What can $W^{\mu\nu}$ depend on? and how must it transform? As it is real and equal to $(W^{\nu\mu})^\dagger$, it must be a real symmetric tensor, and can be a function only of q^μ, p^μ and the scalars W^2 and M^2. By gauge invariance $q_\nu J^\nu = 0$, it must vanish when contracted with q. Noting that

$$W^2 = p'^2 = (p+q)^2 = M^2 + 2p\cdot q + q^2,$$

we may take as the second scalar parameter, instead of W^2,

$$\nu := \frac{p\cdot q}{M} = \frac{Q^2 + W^2 - M^2}{2M}.$$

Thus there are only two form factors available for $W^{\nu\mu}$,

$$W^{\nu\mu}(q,p) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1(Q^2,\nu) + \left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right) \frac{W_2(Q^2,\nu)}{M^2}.$$

We note that there is now a dependence both on Q^2 and ν or W^2, so these form factors are more difficult than the elastic scattering version, where W^2

was fixed at M^2, or $\nu = Q^2/2M$. Comparing to the $e^- X$ scattering, we see that in that case

$$W_2 = Z^2 \delta(\nu - Q^2/2M), \qquad W_1 = \frac{Z^2 Q^2}{4M^2} \delta(\nu - Q^2/2M).$$

The cross section is[4]

$$d\sigma = \frac{1}{2E_A\, 2E_B\, |\vec{v}_A - \vec{v}_B|} \frac{d^3k'}{2\omega'(2\pi)^3} \sum |\mathcal{M}|^2$$

where

$$\sum |\mathcal{M}|^2 = \frac{e^4}{Q^4} 4\pi M W^{\mu\nu} L_{\mu\nu}.$$

Recalling that $q^\mu L_{\mu\nu} = 0$, and $q^2 = (k - k')^2 = -2k \cdot k'$ as we are treating $m_e \sim 0$, we see

$$\begin{aligned} g^{\mu\nu} L_{\mu\nu} &= 4k \cdot k' + 4q^2 = -2Q^2 \\ p^\mu p^\nu L_{\mu\nu} &= 4p \cdot k p \cdot k' - M^2 Q^2. \end{aligned}$$

In the "lab" frame, $p = (M, \vec{0})$, $p \cdot k = Mk$, $p \cdot k' = Mk'$, and $Q^2 = 2k \cdot k' = 2kk'(1 - \cos\theta) = 4kk' \sin^2(\theta/2)$, we have

$$\begin{aligned} \sum |\mathcal{M}|^2 &= \frac{e^4}{Q^4} 4\pi M \left(2Q^2 W_1(Q^2, \nu) + (4kk' - Q^2) W_2(Q^2, \nu) \right) \\ &= \frac{e^4}{Q^4} 4\pi M\, 4kk' \left(2W_1 \sin^2(\theta/2) + W_2 \cos^2(\theta/2) \right). \end{aligned}$$

Then

$$\begin{aligned} \frac{d^2\sigma}{d\Omega dk'} &= \frac{k'}{8(2\pi)^3 kM} \frac{(4\pi\alpha)^2}{(4kk')^2 \sin^4(\theta/2)} 16\pi M kk' \left(2W_1 \sin^2 \frac{\theta}{2} + W_2 \cos^2 \frac{\theta}{2} \right) \\ &= \frac{\alpha^2}{4k^2 \sin^4(\theta/2)} \left(2W_1 \sin^2 \frac{\theta}{2} + W_2 \cos^2 \frac{\theta}{2} \right). \end{aligned}$$

[4] The kinematic initial state factor in the cross section formula can be written in several different ways if we assume the momenta lie along the same axis, say the z axis. The denominator

$$2E_A\, 2E_B |\vec{v}_A - \vec{v}_B| = 4|E_B p_A^3 - E_A p_B^3| = 4|\epsilon_{\mu xy\nu} p_B^\mu p_A^\nu|.$$

This makes it clear that it is invariant under Lorentz boost along the z axis. Furthermore the square is $16(p_A^2 p_B^2 - (p_{A\,\mu} p_B^\mu)^2)$, so the first denominator is $4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}$.

As we have no ϕ dependence, $d\Omega = 2\pi d\cos\theta$, and as $Q^2 = 2kk'(1-\cos\theta)$ and $\nu = k - k'$, and k is fixed, $dQ^2 \wedge d\nu = 2kk' d\cos\theta \wedge dk'$, so

$$\frac{d^2\sigma}{dQ^2\, d\nu} = \frac{\pi\alpha^2}{4k^3k'\sin^4(\theta/2)} \left(2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2} \right).$$