Physics 613 Lecture 10 Feb. 27, 2014

In Lecture 6 we discussed deriving field equations from the lagrangian density, in particular for a scalar field. We observed that the field equations could hold for a real or a complex field, and that a complex field could be considered simply as two real fields with the same mass. We also saw that even though in principle we should vary the two real fields to get the equations of motion, one could get the right equations by varying the complex field $\hat{\phi}$ and its hermitian (or complex) conjugate as if they were independent. The only difference between the real field and the complex field is that the expansion of $\hat{\phi}$ has an independent $b^\dagger(\vec{k})$ rather than $a^\dagger(\vec{k})$, and therefore the conjugate field $\hat{\phi}^\dagger$ contains $a^\dagger(\vec{k})$ and $b(\vec{k})$ and is different from $\hat{\phi}$.

We could have also started with n real scalars with the same mass, with

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{n}\left[\left(\partial_\mu\hat{\phi}_j\right)\partial^\mu\hat{\phi}_j - M^2\hat{\phi}_j^2\right],$$

which would, of course, just give the Klein-Gordon equation for each $\hat{\phi}_j$ independently, and quantization would consist of having operators $a_j(\vec{k})$ and $a_j^\dagger(\vec{k})$, with both operators for different j's commuting with each other.

The Lagrangian is clearly invariant under rotations in the n dimensional space indexed by j, which means under symmetry transformations $\hat{\phi}_j \to \hat{\phi}'_j = \sum_k \mathcal{O}A_{jk}\hat{\phi}_k$, where $\mathcal{O}$ is an orthogonal matrix. The infinitesimal generators of such orthogonal matrices are real antisymmetric matrices A_{jk}. Noether's theorem guarantees us that such a symmetry generates a conserved current. To see this, we note that the variation $\delta\hat{\phi}_j = \sum_k A_{jk}\hat{\phi}_k$ of the lagrangian density is

$$\begin{aligned}
\delta\mathcal{L} &= \sum_{jk} A_{jk}\left[\left(\partial^\mu\hat{\phi}_j\right)\hat{\partial}_\mu\phi_k - M^2\hat{\phi}_j\hat{\phi}_k\right] && \text{which is clearly zero by antisymmetry under } j \leftrightarrow k \\
&= \sum_{jk} A_{jk}\left[\left(\partial^\mu\hat{\phi}_j\right)\hat{\partial}_\mu\phi_k + \hat{\phi}_j\partial^\mu\partial_\mu\hat{\phi}_k\right] && \text{by the equations of motion} \\
&= \partial^\mu\sum_{jk} A_{jk}\hat{\phi}_j\partial_\mu\hat{\phi}_k.
\end{aligned}$$

Thus we see that we can define currents

$$J^\mu_{jk} = \hat{\phi}_j\overleftrightarrow{\partial_\mu}\hat{\phi}_k := \hat{\phi}_j\partial^\mu\hat{\phi}_k - \hat{\phi}_k\partial^\mu\hat{\phi}_j$$

are conserved currents, $\partial_\mu J^\mu_{jk} = 0$. This gives us a global $SO(n)$ symmetry. The conserved charges are, of course,

$$Q_{jk} = \int d^3x J^0_{jk}, \text{ because then } \frac{d}{dt} Q_{jk} = \int d^3x\, \partial_0 J^0_{jk} = \int d^3x\, \vec{\nabla}\cdot\vec{J}_{jk} = \int_S d\vec{S}\cdot\vec{J}_{jk},$$

where S is a sphere at infinity, so assuming the current drops off (more than quadratically) at infinity, the integral is zero, $dQ_{jk}/dt = 0$, and Q_{jk} is conserved.

$SO(n)$ symmetry has played a role in particle physics for several n. $SO(32)$ appears in string theory, and $SO(10)$ has been considered as a candidate grand unified theory (GUT), but by far the most commonly encountered, though usually unrecognized, is $SO(2)$, rotations in a plane. Considering the plane as a complex plane, rotation is just multiplication by a phase, and as we saw in Lecture 6, the complex scalar field fits this description. There is only one conserved current and charge, $J^\mu = J^\mu_{12}$ and $Q = Q_{12}$, with

$$J_\mu = \hat{\phi}_1 \overleftrightarrow{\partial_\mu} \hat{\phi}_2 := \hat{\phi}_1 \partial_\mu \hat{\phi}_2 - \left(\partial_\mu \hat{\phi}_1\right) \hat{\phi}_2$$

with a conserved charge $Q = \int d^3x\, (\pi_2(\vec{x})\phi_1(\vec{x}) - \phi_2(\vec{x})\pi_1(\vec{x}))$. An infinitesimal symmetry transformation is generated by $i\epsilon Q$, with

$$[i\epsilon Q, \phi_1(\vec{y})] = -i\epsilon \int d^3x [\pi_1(\vec{x}), \phi_1(\vec{y})]\phi_2(\vec{x}) = -\epsilon\phi_2(\vec{y}),$$

while $[i\epsilon Q, \phi_2(\vec{y})] = \epsilon\phi_1(\vec{y})$, as given in AH 7.4.

As we discussed above, the Lagrangian density for the $SO(2)$ real scalar field can be written in terms of complex fields[1]

$$\hat{\phi} := \frac{1}{\sqrt{2}}\left(\hat{\phi}_1 - i\hat{\phi}_2\right) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega}} \left[\hat{a}(\vec{k})e^{-ik_\mu x^\mu} + \hat{b}^\dagger(\vec{k})e^{ik_\mu x^\mu}\right], \qquad (1)$$

as $\mathcal{L} = \left(\partial_\mu \phi^\dagger\right)\partial^\mu\phi - M^2\phi^\dagger\phi$. The current becomes $J_\mu = i\hat{\phi}^\dagger \overleftrightarrow{\partial_\mu}\hat{\phi}$. The canonical momentum conjugate to $\hat{\phi}$ is

$$\hat{\pi} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^\dagger = \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega}{2}} \left[i\hat{a}^\dagger(\vec{k})e^{ik_\mu x^\mu} - i\hat{b}(\vec{k})e^{-ik_\mu x^\mu}\right], \qquad (2)$$

[1] Sorry, I don't know why the book defines the complex $\hat{\phi}$ with the minus sign in 7.15, so this rotation is through a negative infinitesimal angle ϵ. I don't think it makes a difference, except that it explains the $e^{-i\alpha}$ in 7.22.

where

$$\hat{a}(\vec{k}) = \frac{1}{\sqrt{2}}\left[\hat{a}_1(\vec{k}) - i\hat{a}_2(\vec{k})\right], \qquad \hat{b}(\vec{k}) = \frac{1}{\sqrt{2}}\left[\hat{a}_1(\vec{k}) + i\hat{a}_2(\vec{k})\right],$$

and then $[\hat{a}_j(\vec{k}), \hat{a}_k^\dagger(\vec{k}')] = (2\pi)^3\delta_{jk}\delta^3(\vec{k} - \vec{k}\,')$ implies

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = (2\pi)^3\delta^3(\vec{k} - \vec{k}\,') = [\hat{b}(\vec{k}), \hat{b}^\dagger(\vec{k}')], \qquad [\hat{a}(\vec{k}), \hat{b}^\dagger(\vec{k}')] = 0,$$

and of course two a's or two b's, without daggers, commute.

In terms of these complex fields, $\hat{Q} = -i\int d^3x \left(\hat{\pi}(\vec{x})\hat{\phi}(\vec{x}) - \hat{\pi}^\dagger(\vec{x})\hat{\phi}^\dagger(\vec{x})\right)$, so $[\hat{Q}, \hat{\phi}(\vec{y})] = -\hat{\phi}(\vec{y})$. This also tells us $[i\epsilon Q, \hat{\phi}] = \delta\hat{\phi}$ for a rotation of the complex $\hat{\phi}$ by an angle of $-\epsilon$.

From the expression for the conserved current $J_\mu = i\hat{\phi}^\dagger\overleftrightarrow{\partial_\mu}\hat{\phi}$, the conserved charge

$$Q = i\int d^3x \left(\hat{\phi}^\dagger\hat{\pi}^\dagger - \hat{\pi}\hat{\phi}\right) = \int \frac{d^3k}{(2\pi)^3}\left[\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) - \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})\right].$$

We see that the $a^\dagger$ creates a positive-charged particle and the $b^\dagger$ creates its negatively-charged antiparticle. The Hamiltonian, of course, is

$$H = \int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}\sum_{j=1}^{2} a_j^\dagger(\vec{k})a_j(\vec{k}) = \int \frac{d^3k}{(2\pi)^3}\omega_{\vec{k}}\left(a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right).$$

The vacuum state $|0\rangle$ is annihilated by all $a_j(\vec{k})$ and by $a(\vec{k})$ and $b(\vec{k})$.

The propagators are given by vacuum expectation values of products of two fields, and as the $\hat{\phi}$'s commute with each other, the only nonzero ones are

$$D_F(\vec{x}_1, \vec{x}_2) = \langle 0|\, T\hat{\phi}(x_1)\hat{\phi}^\dagger(x_2)\,|0\rangle\,.$$

In the evaluation, the b's act just as the a's do, and the same as the a's did in the real scalar case, so the Feynman propagator is the same as it was in that case, with the same Fourier transform,

$$\tilde{D}_F(q) = \frac{i}{q^2 - M^2 + i\epsilon},$$

though now when we claim this represents both the particle going forward in time and the antiparticle going backwards, they are different charges.

Dirac Field

We saw in Lecture 3 that the spin 1/2 Dirac particle obeys an equation of motion $(i\gamma^\mu\partial_\mu - m)\psi = 0$, and that the solutions are complex, so we should anticipate that we will need to have a complex field ψ with 4 complex components, together with its hermitian conjugate $\psi^\dagger$ or equivalently $\bar{\psi} = \psi^\dagger\gamma^0$. We see that we need the conjugate, because the action and the Lagrangian need to be hermitean (in order that $e^{iS/\hbar}$ is unitary). But we see that we can easily get what we want with

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi,$$

because $\partial\mathcal{L}/\partial(\partial_\mu\bar{\psi}) = 0$ and $\partial\mathcal{L}/\partial\bar{\psi} = \gamma^0(i\gamma^\mu\partial_\mu - m)\psi$, so the equations of motion follow from the Euler-Lagrange approach. We know that there are solutions $u(\vec{k}, s)e^{-ik_\mu x^\mu}$ and $v(\vec{k}, s)e^{ik_\mu x^\mu}$ for each 3-momentum $\vec{k}$ and for $s = 1, 2$, so we can write the general solutions as

$$\hat{\psi} = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{s=1}^{2} \left[\hat{c}_s(\vec{k})u(\vec{k}, s)e^{-ik_\mu x^\mu} + \hat{d}^\dagger_s(\vec{k})v(\vec{k}, s)e^{ik_\mu x^\mu}\right],$$

with $k_0 = \omega_k = \sqrt{\vec{k}^2 + m^2}$. Quantization will mean the coefficients $c_s(\vec{k})$ and $d^\dagger_s(\vec{k})$ will become operators, with u and v remaining ordinary complex four-component functions.

It is clear that $\mathcal{L}$ is invariant under a global phase change of ψ, $\psi \to e^{-i\alpha}\psi$, $\bar{\psi} \to \bar{\psi} \to e^{+i\alpha}$, so Noether gauarantees us a conserved current

$$\hat{J}^\mu_\psi = \frac{\partial\mathcal{L}}{\partial\partial_\mu\hat{\psi}}\delta\hat{\psi} = \hat{\bar{\psi}}\gamma^\mu\hat{\psi}, \quad \text{with conserved} \quad Q = \int d^3x\hat{\psi}^\dagger\hat{\psi}.$$

On the other hand, the Hamiltonian is

$$\hat{H} = \int d^3x\hat{\psi}^\dagger\gamma^0\left[-i\vec{\gamma}\cdot\vec{\nabla} + m\right]\psi = i\int d^3x\hat{\psi}^\dagger\frac{\partial}{\partial t}\psi.$$

From the expansions, we have

$$\begin{aligned} Q &= \int d^3x \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{s=1}^{2} \left[\hat{c}^\dagger_s(\vec{k})u^\dagger(\vec{k}, s)e^{ik_\mu x^\mu} + \hat{d}_s(\vec{k})v^\dagger(\vec{k}, s)e^{-ik_\mu x^\mu}\right] \\ &\quad \times \int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_q}} \sum_{s=1}^{2} \left[\hat{c}_s(\vec{q})u(\vec{q}, s)e^{-iq_\mu x^\mu} + \hat{d}^\dagger_s(\vec{q})v(\vec{q}, s)e^{iq_\mu x^\mu}\right] \end{aligned}$$

$$\begin{aligned} = & \int \frac{d^3k}{2\omega_k (2\pi)^3} \sum_{r,s=1}^{2} \Big[\hat{c}_r^\dagger(\vec{k}) \hat{c}_s(\vec{k}) u^\dagger(\vec{k},r) u(\vec{k},s) \\ & + \hat{c}_r^\dagger(\vec{k}) \hat{d}_s^\dagger(-\vec{k}) u^\dagger(\vec{k},r) v(-\vec{k},s) + \hat{d}_r(\vec{k}) \hat{c}_s(-\vec{q}) v^\dagger(\vec{k},r) u(-\vec{q},s) \\ & + \hat{d}_r(\vec{k}) \hat{d}_s^\dagger(\vec{k}) v^\dagger(\vec{k},r) v(\vec{k},s) \Big] \end{aligned}$$

We will need to use

$$\begin{aligned} u^\dagger(\vec{k},r) v(-\vec{k},s) &= v^\dagger(\vec{k},r) u(-\vec{q},s) = 0, \\ u^\dagger(\vec{k},r) u(\vec{k},s) &= 2\omega_k \delta_{rs} = v^\dagger(\vec{k},r) v(\vec{k},s) \end{aligned}$$

so

$$Q = \int \frac{d^3k}{(2\pi)^3} \sum_{s=1}^{2} \left[\hat{c}_s^\dagger(\vec{k}) \hat{c}_s(\vec{k}) + \hat{d}_s(\vec{k}) \hat{d}_s^\dagger(\vec{k}) \right].$$

The hamiltonian calculation is nearly the same, except the $\frac{\partial}{\partial t}$ brings down a ω on each c and a $-\omega$ on each $d^\dagger$, so

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_{s=1}^{2} \left[\hat{c}_s^\dagger(\vec{k}) \hat{c}_s(\vec{k}) - \hat{d}_s(\vec{k}) \hat{d}_s^\dagger(\vec{k}) \right].$$

If these had been reversed, we might have argued that we simply goofed by interchanging the d and $d^\dagger$ in the definition of $\hat{\psi}$, but that would give us a positive definite charge but an energy which could be infinitely negative. The escape is to demand that quantization means that the creation and annihilation operators obey *anti-commutation relations* rather than commutation ones. So

$$\left\{ c_r(\vec{k}), c_s^\dagger(\vec{q}) \right\} = (2\pi)^3 \delta_{rs} \delta^3(\vec{k}-\vec{q}) = \left\{ d_r(\vec{k}), d_s^\dagger(\vec{q}) \right\},$$

$$\left\{ c_r(\vec{k}), c_s(\vec{q}) \right\} = \left\{ c_r(\vec{k}), d_s(\vec{q}) \right\} = \left\{ c_r(\vec{k}), d_s^\dagger(\vec{q}) \right\} = 0.$$

Then, ignoring an infinite constant term, we have

$$Q = \int \frac{d^3k}{(2\pi)^3} \sum_{s=1}^{2} \left[\hat{c}_s^\dagger(\vec{k}) \hat{c}_s(\vec{k}) - \hat{d}_s^\dagger(\vec{k}) \hat{d}_s(\vec{k}) \right].$$

and

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_{s=1}^{2} \left[\hat{c}_s^\dagger(\vec{k}) \hat{c}_s(\vec{k}) + \hat{d}_s^\dagger(\vec{k}) \hat{d}_s(\vec{k}) \right].$$

which is much nicer. Now taking the ground state to satisfy

$$c_r(\vec{k})\,|0\rangle = d_r(\vec{k})\,|0\rangle = 0$$

serves as a base for an arbitrary number of positive energy excitations of particles with positive charge and anti-particles of negative charge (or vice versa if you want to respect Ben Franklin.) The fields now satisfy equal-time anticommutation relations as well,

$$\begin{aligned}
\left\{\hat{\psi}_\alpha(\vec{x},t),\hat{\psi}_\beta(\vec{y},t)\right\} &= 0,\\
\left\{\hat{\psi}_\alpha(\vec{x},t),\hat{\psi}^\dagger_\beta(\vec{y},t)\right\} &= \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta},\\
\left\{\hat{\psi}^\dagger_\alpha(\vec{x},t),\hat{\psi}^\dagger_\beta(\vec{y},t)\right\} &= 0.
\end{aligned}$$

When we get to including interactions and expanding the exponential of the action, we will have nothing new from the time-ordered hamiltonian densities, but in moving the creation and annihilation operators until they annihilate on $\langle 0|$ or $|0\rangle$, we will be *anti-* commuting them through any operators related to the Dirac fields, or more generally any fermionic operators. One can think of "classical" fermionic fields as anti-commuting, and the non-zero right hand side in $\{\hat{\psi},\hat{\psi}^\dagger\}$ as the quantization. Thus it is natural to define the time-ordering meta-operator as

$$T\hat{\psi}(x_1)\bar{\hat{\psi}}(x_2) = \Theta(t_1-t_2)\hat{\psi}(x_1)\bar{\hat{\psi}}(x_2) - \Theta(t_2-t_1)\bar{\hat{\psi}}(x_2)\hat{\psi}(x_1).$$

Note the minus sign.