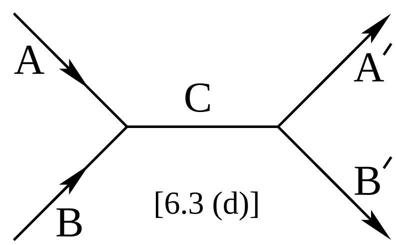
Physics 613 Lecture 9 Feb. 25, 2014

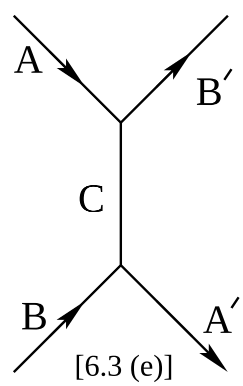
Last time we we looking at the $\mathcal{O}(g^2)$ contribution to the S matrix for the scattering $A + B \to A + B$ in ABC theory, and we found

$$\begin{aligned}\langle \vec{p}^{\,\prime}_A \vec{p}^{\,\prime}_B |\, \hat{S}\, | \vec{p}_A \vec{p}_B \rangle \approx (-ig)^2 \int_0^\infty d^4y \int_{-\infty}^\infty d^4x_2\, D(y, m_C) \\ \times \left[\left(e^{ip'_A\cdot y} + e^{-ip_A\cdot y} \right) e^{i(p'_A - p_A)\cdot x_2} + 2\sqrt{E_A E'_A}(2\pi)^3 \delta^3(p'_A - p_A)\, D(y, m_A) \right] \\ \times \left[\left(e^{ip'_B\cdot y} + e^{-ip_B\cdot y} \right) e^{i(p'_B - p_B)\cdot x_2} + 2\sqrt{E_B E'_B}(2\pi)^3 \delta^3(p'_B - p_B)\, D(y, m_B) \right]\end{aligned}$$



[6.3 (d)]

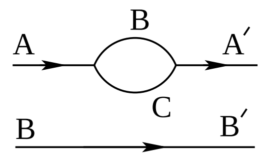
We argued that only the exponential pieces from each square bracket contributed to the scattering amplitude. These four terms each correspond to one of the initial particles spitting off a C particle at x_2, either after having absorbed the other initial particle or going off as one of the final particles. The C particle continues to x_1 and either splits into the two final particles, or scatters off the remaining initial particle. The first possibility in each case corresponds to the diagram 6.3d, where A and B join to form C, which then decays back into $A' + B'$. The second possibility, where either A or B emits a C and turns into B' or A' respectively, and then the C is absorbed by the other incident particle, which turns into the other outgoing particle, is shown in diagram 6.3e. Note this diagram includes the cases where either A or B does the initial emitting, so we don't know which way to consider C is travelling between x_2 and x_1,



[6.3 (e)]

The other terms in the expansion of S all contribute only to the multiplier of $\delta^3(\vec{p}^{\,\prime}_A - \vec{p}_A)\delta^3(\vec{p}^{\,\prime}_B - \vec{p}_B)$, which means no scattering has taken place. Nonetheless they present some new, interesting aspects of evaluation. Consider first the contribution from the $D(y, m_B)$ in the B factor, with the exponential piece from the A factor. This contributes

$$(-ig)^2\sqrt{E_B E'_B}\,2\,(2\pi)^6\delta^3(\vec{p}'_A-\vec{p}_A)\delta^3(\vec{p}'_B-\vec{p}_B)$$
$$\times\int_0^\infty dy^0\int d^3y\left(e^{ip'_A\cdot y}+e^{-ip_A\cdot y}\right)D(y,m_B)D(y,m_C).$$



[6.3 (c)]

This piece is coming from the two fields interacting only with the incoming and outgoing A particles, and corresponds to diagram 6.3c.

As we have a delta function that sets $p'_A = p_A$, the last line can be written

$$\int_{-\infty}^{\infty} d^4y\, e^{ip_A\cdot y}\left(\Theta(y^0)D(y,m_B)D(y,m_C)+\Theta(-y^0)D(-y,m_B)D(-y,m_C)\right)$$

but just as we did for the scattering part (Lecture 8 p 6), the parenthesis can be written in terms of Feynman (time-ordered) propagators $D_F(y,m_B)D_F(y,m_C)$, which we defined last time:

$$D_F(x_1-x_2) := \langle 0|\, T\hat{\phi}(x_1)\hat{\phi}(x_2)\,|0\rangle\,.$$

The Feynman Propagator

We also defined its Fourier transform as $\tilde{D}_F(q^\mu) = \int d^4x e^{iq_\mu x^\mu} D_F(x^\mu)$. We will now show that this has an extremely simple expression.

From the integral for $D(x_1,x_2)$, we see the time-ordered version is

$$D_F(x_1,x_2) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left(\Theta(x_1^0-x_2^0)e^{-ik_\mu(x_1-x_2)^\mu}+\Theta(x_2^0-x_1^0)e^{+ik_\mu(x_1-x_2)^\mu}\right)$$

where k_0 means $\omega_k = +\sqrt{\vec{k}^{\,2}+m^2}$. Taking the Fourier transform,

$$\tilde{D}_F(q) = \int d^4y\, e^{iq_\mu y^\mu}\int\frac{d^3k}{(2\pi)^3 2\omega_k}\left(\Theta(y^0)e^{-i\omega_k y^0}\Theta(-y^0)e^{+i\omega_k y^0}\right)+e^{+i\vec{k}\cdot\vec{y}},$$

where I reversed the sign of the $\vec{k}$ integration variable in the second term. The $\int d^3y$ gives $(2\pi)^3\delta^3(\vec{q}-\vec{k})$, so

$$\tilde{D}_F(q) = \int_{-\infty}^{\infty}\frac{dy^0}{2\omega_{\vec{q}}}\left\{e^{i(q_0-\omega_{\vec{q}})y^0}\Theta(y^0)+e^{i(q_0+\omega_{\vec{q}})y^0}\Theta(-y^0)\right\}.$$

The integral isn't really well defined for real q_0 because of the oscillatory behavior as $q_0\to\pm\infty$, but the first term is well defined for $\operatorname{Im} q_0 > 0$, for which the integral gives $\dfrac{1}{2\omega_{\vec q}}\dfrac{i}{q_0-\omega_{\vec q}}$, and the second part is well defined for Im $q_0 < 0$, giving $\dfrac{1}{2\omega_{\vec q}}\dfrac{-i}{q_0+\omega_{\vec q}}$. Throw in an infinitesimal piece to pretend that real q_0 has the required imaginary part, and we have

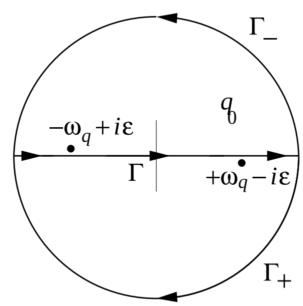
$$\tilde{D}_F(q) = \frac{1}{2\omega_{\vec q}}\left(\frac{i}{q_0-\omega_{\vec q}+i\epsilon} - \frac{i}{q_0+\omega_{\vec q}-i\epsilon}\right) = \frac{i}{q_0^2-\vec q^{\,2}-m^2+i\epsilon} = \frac{i}{q^2-m^2+i\epsilon}$$

where ϵ is an infinitesimal positive quantity.

This hocus-pocus with the ϵ could probably use a bit more explication. If we undo the Fourier transform,

$$\begin{aligned} D_F(x_1,x_2) &= \int \frac{d^4q}{(2\pi)^4}\, e^{-iq_\mu(x_1-x_2)^\mu}\,\tilde{D}_F(q) \\ &= \frac{1}{(2\pi)^4}\int d^3\vec q\, e^{i\vec q\cdot(\vec x_1-\vec x_2)}\int \frac{dq_0\, e^{-iq_0(t_1-t_2)}}{2\omega_{\vec q}}\left\{\frac{i}{q_0-\omega_{\vec q}+i\epsilon} - \frac{i}{q_0+\omega_{\vec q}-i\epsilon}\right\} \end{aligned}$$

The integral over real q_0 along the contour Γ can be evaluated by closing the contour with an infinite semicircle in the complex plane, providing the contribution from that contour vanishes. If $t_1 > t_2$, the exponential goes to zero for large negative Im q_0, so we can close the contour with Γ_+. Then we just get $-2\pi i$ times the residue of the pole at $+\omega_{\vec q}$, so $2\pi e^{-i\omega_{vecq}(t_1-t_2)}$, and the full propagator



$$D_F(x_1-x_2) = \int \frac{d^3q}{(2\pi)^3}\, e^{-iq_\mu(x_1-x_2)^\mu}\Big|_{q_0=\omega_{\vec q}} \qquad \text{for } t_1 > t_2.$$

On the other hand, if $t_1 < t_2$, the exponential vanishes for the large semicircle in the upper half, so now the contour includes only the pole at $-\omega_{\vec q}$, goes in the stardard counterclockwise direction, so we get $2\pi e^{+i\omega_{\vec q}(t_1-t_2)}$. Changing the sign of the integration variable $\vec q$, we have

$$D_F(x_1-x_2) = \int \frac{d^3q}{(2\pi)^3}\, e^{-iq_\mu(x_2-x_1)^\mu}\Big|_{q_0=\omega_{\vec q}} \qquad \text{for } t_2 > t_1.$$

But we see that in both cases this is the time ordered propagator,

$$\langle 0|\, T\hat\phi(y)\hat\phi(0)\,|0\rangle\,.$$

Return to Fig. 6.3(c)

Let us return to the evaluation of the contribution to $\hat{S}$ given by Figure 6.3(c), or rather to the top piece, which is a transform of a product of two Feynman propagators:

$$\begin{aligned}
&\int_{-\infty}^{\infty} d^4y\, e^{ip_A\cdot y} D_F(y,m_B) D_F(y,m_C) \\
&= \frac{1}{(2\pi)^8}\int_{-\infty}^{\infty} d^4y\, e^{ip_A\cdot y}\int_{-\infty}^{\infty} d^4q\, e^{-iq_\mu y^\mu}\tilde{D}_F(q,m_B)\int_{-\infty}^{\infty} d^4k\, e^{-ik_\mu y^\mu}\tilde{D}_F(k,m_C) \\
&= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4q\int_{-\infty}^{\infty} d^4k\, \delta^4(p_A-q-k)\,\tilde{D}_F(q,m_B)\tilde{D}_F(k,m_C) \\
&= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4q\,\tilde{D}_F(q,m_B)\tilde{D}_F(p_A-q,m_C) \\
&= \frac{1}{(2\pi)^4}\,\frac{i}{q^2-m_B^2+i\epsilon}\,\frac{i}{(p_A-q)^2-m_C^2+i\epsilon}.
\end{aligned}$$

At this point we note first, that unlike the scattering pieces, in which the momentum of the propagator $D_F(q)$ was at definite momenta, either p_A+p_B or p'_A-p_B, here we are left with an integral over the momentum q.

If we examine the expressions we found from contracting the $\hat{\phi}$ fields and the creation and annihilation operators from the initial states, we see that each factor of the hamiltonian density corresponds to a vertex, or point where lines join, and each contraction corresponds either to an external line connecting to another or to an internal vertex, or to a line which connects two vertices. In momentum space, each vertex gives a factor of $-ig$, each internal line gives a factor of $\dfrac{i}{q_j^2-m_j+i\epsilon}$, where q_j^μ and m_j are the momentum and mass associated with the line, there is momentum conservation at every vertex, and any internal line's momentum not determined by momentum conservation is to be integrated over, $\int d^4q/(2\pi)^4$. We will get to see that this is general, although there may also be symmetry factors $1/S$ which we will get to when we consider retrospectively and then more generally what we did to get these expressions for the matrix elements S_{fi}. But first, lets discuss the scattering cross section for colliding A and B beams.

At the end of lecture 8 we found that

$$\begin{aligned}
\langle \vec{p}\,'_A\vec{p}\,'_B|\,\hat{S}\,|\vec{p}_A\vec{p}_B\rangle \;\approx\; &-g^2(2\pi)^4\delta\left(p'_B-p_B+p'_A-p_A\right) \\
&\times\left[\tilde{D}_F(p_A+p_B,m_C)+\tilde{D}_F(p'_A-p_B,m_C)\right].
\end{aligned}$$

which means the invariant amplitude is

$$\begin{aligned} i\mathcal{M} &= -g^2\left[\tilde{D}_F(p_A+p_B,m_C)+\tilde{D}_F(p'_A-p_B,m_C)\right] \\ &= -ig^2\frac{1}{(p_A+p_B)^2-m_C^2}-ig^2\frac{1}{(p'_A-p_B)^2-m_C^2}. \end{aligned}$$

Notice the first denominator is $s-m_C^2$, where s is the square of the center of mass energy, while the denominator of the second term is $u-m_c^2$, where u is the square of the 4-momentum transfer from the incoming B particle to the outgoing A particle. At one time there was a lot of interest in scattering with just two particles in the final state, and the lorentz-invariant part of such scattering is well describe by *Mandelstam variables* s, t and u, where t is $(p'_A-p_A)^2$. Of course more generally the particles A' and B' can be different types that the initial A and B, so which to call A', and hence t is the momentum transfer, is a matter of convention, which is, if possible, to choose the outgoing particle most similar to A. If we call the outgoing particles C and D, we have

$$\begin{aligned} s &= (p_A+p_B)^2=m_A^2+m_B^2+2p_{A\,\mu}p_B^\mu=(p_C+p_D)^2=m_C^2+m_D^2+2p_{C\,\mu}p_D^\mu \\ t &= (p_C-p_A)^2=m_A^2+m_C^2-2p_{A\,\mu}p_C^\mu=(p_B-p_D)^2=m_B^2+m_D^2-2p_{B\,\mu}p_D^\mu \\ u &= (p_D-p_A)^2=m_A^2+m_D^2-2p_{A\,\mu}p_D^\mu=(p_B-p_C)^2=m_B^2+m_C^2-2p_{B\,\mu}p_C^\mu \end{aligned}$$

adding,

$$\begin{aligned} s+t+u &= 3m_A^2+m_B^2+m_C^2+m_D^2+2p_{A\,\mu}(p_B-p_C-p_D)^\mu \\ &= m_A^2+m_B^2+m_C^2+m_D^2, \end{aligned}$$

because $p_B-p_C-p_D=-p_A$ by momentum conservation. Thus for a fixed set of particles, there are only two scalar parameters needed to describe the kinematic factors in the invariant amplitude.

For the cross section, however, we need the noninvariant factor $4E_AE_B\,|\vec{v}_A-\vec{v}_B|$ and to express the final momentum integrals in terms of scattering angles. If we work in a frame where the initial particles move along a common line, say in the z direction, the non-invariant factor $4E_AE_B|\vec{v}_A-\vec{v}_B|=4(p_A^3p_B^0-p_B^3p_A^0)=4\epsilon_{\mu\nu23}p_A^\mu p_B^\nu$, which shows it is Lorentz invariant for boost in the z direction. So we can easily translate from the lab to the center-of-mass frame.

In the center of mass frame, $p_A^3 = -p_B^3 = p$, $v_A = p/E_A$ so $4E_A E_B|\vec{v}_A - \vec{v}_B| = 4pE_{\rm cm}$ and

$$\left(\prod_{f=1}^{n} \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\,\delta^4\left(k_A^\mu + k_B^\mu - \sum_{j=1}^{n} p_j^\mu\right)$$
$$\to \qquad \frac{p'^2 dp'\, d\cos\theta\, d\phi}{16\pi^2 E_C E_D}\delta(E_C + E_D - E_{\rm cm})$$

As for the particle decay calculation, the delta function fixes p' to be given by 6.65 with m_C replaced by $\sqrt{s}$ and m_A and m_B replaced by m_C and m_D, and we can replace $dp'\delta(E_C + E_D - E_{\rm cm})$ by $E_C E_D/p' E_{\rm cm}$. Thus all together,

$$\frac{d\sigma}{d\Omega} = \frac{1}{4pE_{\rm cm}}\frac{p'}{16\pi^2 E_{\rm cm}}\left|\mathcal{M}(s,u)\right|^2,$$

where $d\Omega = \sin\theta d\theta d\phi$ as usual.

In our case, the particles in the initial state are the same as in the final state, so in the center-of-mass $p = p'$, and we have

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm cm} = \frac{1}{64\pi^2 E_{\rm cm}^2}\left|\mathcal{M}(s,u)\right|^2.$$

We also have $t = (p'_A - p_A)^2 = 2m_A^2 - 2(E_A^2 - p^2\cos\theta)$ so $u = 2m_A^2 + 2m_B^2 - s - t = 2m_B^2 - s + 2(E_A^2 - p^2\cos\theta)$. For simplicity, let's assume $m_A = m_B$, so $E_A^2 = s/4$, $p^2 = E_A^2 - m_A^2 = \frac{s}{4} - m_A^2$, $u = -2p^2(1 + \cos\theta)$.?? In this case,

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm cm} = \frac{g^4}{64\pi^2 s}\left(\frac{1}{s - m_C^2} - \frac{1}{(s - 4m_A^2)\cos^2(\theta/2) + m_C^2}\right)^2.$$

There are some comments in order here.

- Notice that the two Feynman diagrams give additive contributions to the **amplitude**, and therefore there are interference terms in the cross section.

- The smallest possible value of s is $m_A + m_B$, but if that is less than m_C, our calculation gives an infinite cross section when $s = m_C$. However, when this is the case C is not a stable particle, and in some ways we might consider that the mass has an imaginary part, and the amplitude should be $1/(s - m_C + i\Gamma/2$, where Γ is the decay width we previously calculated.

- If type C particles are massless, the second term has a vanishing denominator when the scattering angle $\theta = \pi$, when the C propagator has zero momentum, and the B' particle comes off with the initial momentum of the A particle. This is similar to what happens with photon exchange, except we don't switch A and B there. This leads to a cross section that blows up as $\sin^4((\pi - \theta)/2)$, reminiscent of the Rutherford forward scattering cross section.

- If type C particles are not massless, but $m_C \ll s$, the second term will still dominate, though not blow up. This will lead to a Yukawa type potential (ignoring the $A \leftrightarrow B$ switch) as explained in Chapter 1 (note Eq. 1.16 looks like $p^2 - 1/a^2$ for a propagator).