An extra note on Noether's Theorem

I was asked to clarify the connection of conserved charges emerging as a consequence of a symmetry and the generator of those symmetry transformations.

Let us consider the simpler case of a discrete dynamical system, with coordinates q_i, with a Lagrangian $L(q_i, \dot{q}_i, t)$ which is invariant under an infinitesimal symmetry transformation $q_i \to q_i + \epsilon_i(q,t)$, so that the change produced in the Lagrangian is

$$\delta L = \sum_i \left[\frac{\partial L}{\partial q_i} \epsilon_i(q,t) + \frac{\partial L}{\partial \dot{q}_i} \frac{d\epsilon_i(q,t)}{dt} \right]$$

where the time derivative of ϵ is a stream derivative including the variation of all of the q's, $\dfrac{d\epsilon_i(q,t)}{dt} = \dfrac{\partial \epsilon_i(q,t)}{\partial t} + \sum_j \dfrac{\partial \epsilon_i(q,t)}{\partial q_j} \dot{q}_j$. Using the equations of motion on the first term in δL, we have

$$\delta L = \sum_i \left[\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \epsilon_i(q,t) + \frac{\partial L}{\partial \dot{q}_i} \frac{d\epsilon_i(q,t)}{dt} \right] = \frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \epsilon_i(q,t) \right).$$

Thus if the change δL is zero,

$$Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} \epsilon_i(q,t) = \sum_i P_i \epsilon_i(q,t)$$

is conserved. Here P_i is the canonical momentum conjugate to q_i.

But we see that the quantum mechanical commutator or classical Poisson bracket generates (minus) the infinitesimal symmetry transformation,

$$\begin{aligned} [Q, q_i] &= \left[\sum_j P_j \epsilon_j(q,t), q_i \right] = \sum_j [P_j, q_i] \, \epsilon_j(q,t) = \sum_j [-\delta_{ij}] \, \epsilon_j(q,t) \\ &= -\epsilon_i(q,t) = -\delta q_i \end{aligned}$$

In Field Theory

In field theory the situation is similar but complicated by our consideration of changes in the x^μ as well as in the fields.

We considered an infinitesimal variation

$$x_\mu \to x'_\mu = x_\mu + \delta x_\mu \tag{1}$$

with the fields varying by

$$\phi'_i(x') = \phi_i(x) + \delta\phi_i(x; \phi_k(x)) = \phi_i(x') + \mathfrak{d}\phi_i. \tag{2}$$

The two forms $\delta\phi$ and $\mathfrak{d}\phi$ are useful in differing expressions for $\delta\mathcal{L}$, which we defined by

$$\delta\mathcal{L}(\phi'_i(x'), \partial'_\mu \phi'_i(x'), x') = \mathcal{L}(\phi'_i(x'), \partial'_\mu \phi'_i(x'), x') - \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x) \left| \frac{\partial x^\nu}{\partial x'^\mu} \right|.$$

We found that the current for this infinitesimal transformation

$$\epsilon J^\mu = -\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial_\nu \phi_i \delta x^\nu - \mathcal{L} \delta x^\nu + \Lambda^\mu. \tag{3}$$

has a vanishing divergence if $\delta\mathcal{L} = \partial_\mu \Lambda^\mu$.

Let us avoid some interpretation problems by assuming the transformation doesn't change time $\delta t = 0$ so we have no problem defining a conserved charge as $Q = \int d^3x J^0(\vec{x})$, and let's assume we don't need a Λ. Then

$$Q = -\int d^3x \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}(\vec{x}) \left(\delta\phi_i(\vec{x}) + \delta x^\nu \partial_\nu \phi_i(\vec{x}) \right) = -\int d^3x \pi_i(\vec{x}) \mathfrak{d}\phi_i(\vec{x}),$$

where $\pi_i(\vec{x})$ is the canonical momentum density conjugate to ϕ_i. Then

$$\begin{aligned} [Q, \phi_j(\vec{y})] &= -\int d^3x \left[\pi_i(\vec{x}) \mathfrak{d}\phi_i(\vec{x}), \phi_j(\vec{y}) \right] = -\int d^3x \left[\pi_i(\vec{x}), \phi_j(\vec{y}) \right] \mathfrak{d}\phi_i(\vec{x}) \\ &= -\int d^3x \left[-i\delta^3(\vec{x} - \vec{y}) \delta_{ij} \right] \mathfrak{d}\phi_i(\vec{x}) = i \mathfrak{d}\phi_j(\vec{y}). \end{aligned}$$

So Q does generate the infinitesimal symmetry transformation at each point in space.